HANDEL-MILLER THEORY AND FINITE DEPTH FOLIATIONS

JOHN CANTWELL AND LAWRENCE CONLON

ABSTRACT. We make a detailed study of the unpublished work of M. Handel and R. Miller on the classification, up to isotopy, of endperiodic homeomorphisms of surfaces. We generalize this theory to surfaces with infinitely many ends and we axiomatize the theory for the important case in which the laminations cannot be assumed to be geodesic. We set the stage for projected applications to foliations of depth greater than one. Using the axioms, we provide a completely new proof of the so-called "transfer theorem", namely that, if two depth one foliations $\mathcal{F}$ and $\mathcal{F}'$ are transverse to a common one-dimensional foliation $\mathcal{L}$ which induces Handel-Miller monodromy on the noncompact leaves of $\mathcal{F}$, then $\mathcal{L}$ also induces Handel-Miller monodromy on the noncompact leaves of $\mathcal{F}'$. An earlier published proof was very complicated and contained some errors. Finally, the axioms also let us smooth the Handel-Miller map.

1. Introduction

The Nielsen-Thurston theory of homeomorphisms of compact surfaces [2, 28, 23, 24] classifies the isotopy class of an irreducible, nonperiodic automorphism f of a compact, hyperbolic surface. There is a pair of mutually transverse geodesic laminations that arise canonically from f and a homeomorphism h, isotopic to f, that preserves these laminations. The automorphism h is expanding on one of the laminations and contracting on the other and is unique on their intersection. Such an automorphism is called "pseudo-Anosov". Following this lead, the Handel-Miller theory [22] considers irreducible endperiodic homeomorphisms f of noncompact surfaces with finitely many ends, none of which is simple (see Definition 2.1). In this case, the analogue of the periodic case is a total translation. In the "pseudo-Anosov" case, the classification, up to isotopy, again proceeds by determining a pair of mutually transverse geodesic laminations and finding an endperiodic representative h of the isotopy class of f that preserves these laminations.

An endperiodic homeomorphism f arises naturally as the monodromy map on a depth one leaf L in a transversely orientable, taut, depth one foliation of a compact 3-manifold M. If M is not orientable, L may or may not be orientable and, if L is orientable, f will be orientation reversing. The Handel-Miller theory figures prominently in the work of S. Fenley on such foliations [14, 15], and elsewhere [10], but a complete account has never been published (see, however, [15, Section 2] for a summary).

The Handel-Miller theory also applies to the monodromy of a locally stable leaf of higher depth in a transversely orientable, taut, finite depth foliation of class C^2, an interesting fact that has not been remarked elsewhere. Such a leaf L, if at depth k, has topological type $k-1$ (Definition 2.2) and has no simple ends if there are

1991 *Mathematics Subject Classification.* Primary 37E30; Secondary 57R30.
Key words and phrases. endperiodic, lamination, finite depth.

no leaves that are annuli, tori, Möbius strips nor Klein bottles. Its monodromy f is endperiodic (Definition 2.10). The "interesting" dynamics of f, however, occurs in an f-invariant subsurface $L' \subset L$ of type 0 (the "soul" of L), obtained by paring away excess (nonperiodic) ends of L. The map $f|L'$ can then be analyzed by the Handel-Miller theory.

In [10], the Handel-Miller theory was used to give a cohomological classification, up to isotopy, of taut depth one foliations. In a paper [5] which is in preparation, we generalize [10] to stable components of finite depth foliations of class C^2 and even to those of class C^0, while filling in some gaps and simplifying many proofs. Here, the generalization of Handel-Miller to surfaces of higher type plays a key role.

After establishing the reduction to type 0 surfaces, we will endeavor to give a complete account of the Handel-Miller theory. In our account, we identify the reducing curves late in the game, first constructing and analyzing the laminations. Standard treatments of Thurston's classification theory for compact surface automorphisms [2, 13, 23, 24] begin by cutting the surface apart along a finite family of reducing circles and then proving that the automorphism in each piece is either periodic or pseudo-Anosov. Fenley [15] outlines a similar approach in the endperiodic case. In the compact case, our approach would also be possible, but a bit awkward, first producing a pair of transverse (non-minimal) geodesic laminations and then, after identifying certain "crown sets" in the complements of these laminations [2, Lemma 4.4], finding the reducing circles, the pseudo-Anosov pieces and the periodic pieces. In the case of endperiodic automorphisms, we find our approach–postponing the reduction–the more attractive one. In this case, the reducing curves need not be compact and even the compact ones may not come from crown sets. By producing the laminations first, we find that the reducing curves show up in very natural ways. Those that do not come from crown sets show up in "escaping" regions complementary to the laminations, something for which there is no analogue in the compact case. We feel that this approach gives added insight and it will enable us to eliminate the somewhat cumbersome Section 3 of [10] in our projected revision and generalization of that paper [5]. Another advantage is that, after reducing, there may be pieces that have simple ends, something that we exclude for L itself since, when L is a leaf in a compact foliated manifold, simple ends prevent the natural metric induced on L from being quasi-isometric to a hyperbolic metric. And there might also be trivial pieces that are "total translations" which we exclude for L itself. The major disadvantage to our approach is that the desired lamination-preserving, endperiodic homeomorphism h, isotopic to the original f, cannot be defined on all of L before reduction, but only on the laminations. This causes some technical difficulties, but no essential problems.

As one of our primary goals, we will prove a key result, Theorem 11.1, the proof of which in [10, Theorem 5.8] had errors, glossed over some delicate points and was much too complicated. In its most general form, this result says that, for a stable component $W \subset M$ in a finite depth foliation $\mathcal{F}$, a transverse, leaf-preserving flow on W inducing the Handel-Miller automorphism on each leaf of $\mathcal{F}|W$ likewise induces the Handel-Miller automorphism on the leaves of any stable refoliation $\mathcal{F}'|W$ that is also transverse to the flow. In this theorem, the Handel-Miller laminations on the leaves of $\mathcal{F}'|W$ cannot be assumed to be geodesic. In Subsection 4.3, we give a partial axiomatization of the nongeodesic version of Handel-Miller, thereafter developing the entire theory on the basis of these axioms. In Section 10, we give

a revised and complete set of axioms. The proof of Theorem 11.1 then proceeds by showing that the the truth of these axioms for the monodromy of $\mathcal{F}|W$ implies their truth for that of $\mathcal{F}'|W$.

We close with a proof that the Handel-Miller monodromy can be chosen to be a diffeomorphism, except at finitely many p-pronged singularities (Theorem 12.1) where the lack of differentiability is well-understood and is a minor issue. This result is needed for applications of the Schwartzmann-Sullivan theory of asymptotic cycles [26, 27] in our above-cited paper [5].

The reader might find it helpful, while reading this paper, to consult the examples of endperiodic maps and Handel-Miller laminations sketched in [11].

We thank Sergio Fenley for many useful conversations about endperiodic maps for surfaces of type 0. He gives some interesting examples of endperiodic maps in [14] and some theorems beyond the Handel-Miller theory in [15, Section 3]. He has also looked at parts of a preliminary draft of this paper and made many helpful comments and suggestions.

2. Endperiodic Maps and Foliations

2.1. **Basic Endperiodic Theory.** Let L be a noncompact, connected surface, possibly with boundary, and let $f : L \to L$ be a homeomorphism. We do not assume that L is orientable nor, if it is, that f is orientation preserving.

The following term will come up frequently throughout this paper.

Definition 2.1. An end of L is simple if it has a neighborhood homeomorphic to $S^1 \times [0, \infty)$ or $[0, 1] \times [0, \infty)$.

Definition 2.2. The surface L has *topological type* $k \geq 0$ if the k^{th} derived set of its endset is finite and non-empty. The elements of this set are called ends of type k and, more generally, an end is of type j if it is isolated in the j^{th} derived endset. If L is compact it is said to have topological type -1.

We assume that L is of topological type $k \geq 0$.

Notice that f induces an automorphism on the space of ends of L which, by abuse, we will also denote by f.

Definition 2.3. An end e of L is *cyclic* of *period* $p \geq 1$ if $f^p(e) = e$ and p is the least such integer.

Definition 2.4. An end e is *attracting* (or a *positive end*) if there is a neighborhood U of e and an integer $n > 0$ such that $f^n(\overline{U}) \subset U$ and $\bigcap_{k=1}^{\infty} f^{kn}(U) = \emptyset$. The end is *repelling* (or a *negative end*) if the above holds for some integer $n < 0$. Attracting and repelling ends are called *periodic ends*.

The following is obvious.

Lemma 2.5. *An end e is an attracting* (*or positive*) *end iff it has a fundamental system of open neighborhoods $U_0 \supset U_1 \supset U_2 \supset \dots$, with $U_i \supset \overline{U}_{i+1}$, such that $f^{n_e}(U_i) = U_{i+1}$, $0 \leq i < \infty$, for some integer $n_e > 0$. Similarly for repelling* (*negative*) *ends with $n_e < 0$.*

Here and throughout this paper, the inclusion symbols "$\supset$" and "$\subset$" will denote proper inclusions. Otherwise, we will use "$\supseteq$" and "$\subseteq$".

Remark. The notation "n_e" introduced above will be used throughout this paper.

For simplicity of exposition, we often consider only attracting ends explicitly in this section and trust the reader to adapt the discussion to repelling ends.

Evidently, a periodic end e is cyclic with period dividing n_e, but cyclic ends may fail to be periodic. For instance, on a four times punctured sphere, one easily produces a homeomorphism f that cyclically permutes three punctures, fixes the third and satisfies $f^3 = \mathrm{id}$. So these ends are cyclic but not periodic. Similar examples can be produced with an arbitrary number $k \geq 3$ of ends – and examples in which the ends are non-planar.

By the standard definition of neighborhood of an end e, each $\overline{U}_i$ is a noncompact, connected surface, separated from the rest of L by the compact 1-manifold which is the closure of $\partial\overline{U}_i \smallsetminus \partial L$. Thus, $B_i = \overline{U}_i \smallsetminus U_{i+1}$ is a (generally noncompact) closed subsurface with the closure of $\partial B_i \smallsetminus \partial L$ compact, $0 \leq i < \infty$. Furthermore,

$$f^{n_e}(B_i) = B_{i+1}, 0 \leq i < \infty,$$

and so these surfaces are mutually homeomorphic and $\overline{U}_i = B_i \cup B_{i+1} \cup \cdots$, $\forall i \geq 0$.

Definition 2.6. The surfaces B_i are called fundamental domains for the attracting end e.

Definition 2.7. The compact 1-manifolds $J_i = \overline{U}_i \smallsetminus U_i$, $1 \leq i < \infty$, are called the positive junctures. The corresponding 1-manifolds for repelling ends are called the negative junctures.

Note that ∂B_i consists of $J_i \cup J_{i+1}$ along with pieces of ∂L. The junctures need not be connected and are mutually homeomorphic via f. It should be remarked also that the fundamental domains and junctures are highly non-canonical.

Lemma 2.8. *Without loss of generality, it may be assumed that the fundamental domains are connected.*

Proof. Suppose that B_0 has more than one connected component. Since U_0 is connected, we find finitely many paths $s_1, \dots, s_r$ in U_0 that connect the components of B_0. (By the definition of neighborhood of an end, ∂B_0 is compact, hence has finitely many components.) That is $B_0 \cup s_1 \cup \cdots \cup s_r$ is connected. For a minimal integer $m \geq 1$, these paths all lie in $B = B_0 \cup B_1 \cup \cdots \cup B_m$ and we claim that B is connected and can be taken as a new choice of fundamental domain B_0 for f^{n+m+1}. Indeed, each component of B_1 attaches to B_0 along at least one component of J_1, and so $B_0 \cup B_1 \cup s_1 \cup \cdots \cup s_r \subset B$ is connected. Repeating this reasoning finitely often, we obtain the claim. $\square$

Lemma 2.9. *Without loss of generality, it may be assumed either that each juncture is a finite collection of simple closed curves, or a finite collection of compact arcs with endpoints on ∂L. If all boundary components of L are compact, the components of the junctures may be assumed to be simple closed curves.*

Proof. If the juncture J_0 is a nonempty collection of simple closed curves and compact arcs, the simple "tunneling" procedure described in [4, page 132] adapts to our situation to to produce from a simple closed curve and an arc a compact arc. Here, one wants the assumption, guaranteed by Lemma 2.8, that B_0 is connected. If all components of J_0 are simple closed curves, the same procedure can be used to connect them to be a single simple closed curve, but for technical reasons we prefer not to do this. Iterations of f^{n_e} propagate this structure to such a choice for all

of the junctures for the periodic end. If all components of ∂L are circles, then the only way that J_0 can separate L is that, whenever some component of J_0 meets a circle component $C \subseteq \partial L$ in a point x, then some (possibly different) component of J_0 meets C in a point $y \neq x$. Slightly displacing an arc of C joining x, y, we coalesce these components of J_0 to a single closed one. After finitely many steps we obtain a juncture that replaces J_0 and has only circle components. □

Remark. If an isolated periodic end is not simple and its junctures are collections of circles, then its fundamental domains B_i have negative Euler characteristic. If the junctures are unions of arcs, $B_i \cup B_{i+1} \cup \cdots \cup B_{i+r}$ has negative Euler characteristic, for some integer r. By increasing the exponent n_e of f^{n_e}, we can assume that each fundamental domain has negative Euler characteristic. This fact will have important uses.

Definition 2.10. The map $f : L \to L$ is called an *endperiodic homeomorphism* of L if all cyclic ends are periodic.

Since there are only finitely many ends of type k, these ends are all cyclic, hence they are periodic. We prove some elementary facts about endperiodic homeomorphisms.

Proposition 2.11. *If $k \geq 1$, an endperiodic homeomorphism has both attracting and repelling ends of type k. If $k = 0$ and L has at least one nonsimple end, the same assertion holds.*

Proof. Suppose that L has repelling ends of type $k \geq 1$, each with a fundamental neighborhood system $U_0 \supset U_1 \supset U_2 \supset \dots$ as above. Let V_0 be the union of all the U_0 for all the repelling ends of type k and let $V = \displaystyle\bigcup_{i=1}^{\infty} f^i(V_0)$. Then $V \smallsetminus V_0$ contains neighborhoods of infinitely many ends of type $k-1$ not in a fundamental neighborhood U_0 for a repelling end of type k. These ends must accumulate on an end of type k, necessarily an attracting end. Similarly, if there are attracting ends of type k there are repelling ends of type k.

Suppose that L is of type 0 and has all ends repelling, at least one of which, say e, is nonsimple. That is, every fundamental domain B_i (compact in this case) corresponding to a fundamental neighborhood system of e can be assumed to have negative Euler characteristic. Since all ends are repelling, the family $\{f^{qn}(B_0)\}_{q=0}^{\infty}$ is a family of compact surfaces with disjoint interiors, each of negative Euler characteristic, which does not accumulate at any end. This gives the contradiction that some compact subsurface $S \subset L$ has infinite Euler characteristic. If all ends are attracting, one uses the same argument, replacing f with f^{-1}. □

Proposition 2.12. *If $f : L \to L$ is endperiodic, there are only finitely many periodic ends.*

Proof. We first remark that if e is a periodic end, then e cannot lie in a fundamental neighborhood U_0 for another periodic end e'. For then, a fundamental domain B_i for e' is a neighborhood of e for some $0 \leq i < \infty$, and so the iterates $f^i(e)$ would all have to be distinct, contradicting the fact that e is cyclic. If there are infinitely many periodic ends, they accumulate on some end e_0 which, as above, cannot be periodic. Therefore the $e_i = f^i(e_0)$ are all distinct. Since periodic ends accumulate on e_0, by continuity and induction periodic ends accumulate on the e_i, $0 \leq i < \infty$.

Since the e_i accumulate on e_0' of one higher type, we eventually get periodic ends accumulating on an end of type k, which, as we know, is periodic. This contradiction completes the proof. $\square$

Remark. One might think that at most finitely many noncompact boundary components can issue from a given end of L. But see [4, Figure 12.5.11] for a surface with one end e and infinitely many noncompact boundary components issuing from and terminating at e. Of course, this surface does not admit an endperiodic map, but using infinitely many connected sums, one easily concocts an endperiodic map on a surface with infinitely many isolated nonperiodic ends of this type. Thus, the periodicity hypothesis in the following lemma is essential.

Lemma 2.13. *If $f : L \to L$ is endperiodic, then only finitely many noncompact boundary components can issue from a periodic end and these must terminate at a periodic end.*

Proof. Let e be a periodicg end and consider its junctures $\{J_i\}_{i=1}^{\infty}$. By Lemma 2.9, either each J_i is a collection of simple closed curves and no noncompact boundary component issues from e, or each J_i consists of the same finite number of compact arcs. Any noncompact component of ∂L issuing from e must meet some J_i in an endpoint of an arc component, hence meets every J_{i+r}, $r \geq 0$ in an endpoint. If the number of noncompact boundary components issuing from e were infinite, a large enough choice of r would produce a juncture J_{i+r} meeting more such boundary components than the number of available endpoints.

Now f^n permutes the finitely many boundary components that issue from periodic ends, hence permutes the finitely many ends at which they terminate. These ends, being cyclic, are periodic. $\square$

Remark. It should be noted that it is possible that a component of ∂L might be a line ℓ joining two positive ends or two negative ones. In that event, f will have at least one periodic point on ℓ. It will turn out that ℓ will then be a leaf of one of the two transverse laminations to be constructed in Section 4, while a ray issuing from the periodic point will be a leaf of the other. This causes some difficulties for us, but the problem can be eliminated without loss of generality by doubling along all such components ℓ. One doubles the endperiodic map f, obtaining an endperiodic map $2f$. The original surface remains as a $2f$-invariant subsurface in $2L$. The remaining noncompact boundary components which issue from periodic ends will join negative ends to positive ones and it is easy to modify the endperiodic map near these boundary lines so that there are no periodic points on them. Accordingly, we assume henceforth that **each noncompact component of ∂L, issuing from a periodic end, joins a negative end to a positive end and contains no f-periodic point**.

Finally, we turn to the process of "paring off" the nonperiodic ends of L to reduce the "interesting" dynamics of f to its action on an f-invariant subsurface $L' \subset L$ of type 0, the Handel-Miller situation. This process is an example of what we will later call a "reduction" of the endperiodic surface. We assume that L has no simple ends.

Consider an end e of type k, necessarily a periodic end which, for definiteness, we assume to be attracting. The fundamental domain B_0 is a neighborhood of finitely many ends $\{e^1, \dots, e^r\}$ of type $k-1$. For each j, $1 \leq j \leq r$, the sequence

$\{f^q(e^j)\}_{q=-\infty}^{\infty}$ clusters at some ends of type k. All but finitely many of these ends of type $k-1$ has a neighborhood that is a fundamental domain for one or another of these ends of type k, and so e^j has a closed neighborhood V^j which is carried by all but finitely many f^q into these fundamental domains. Excising all $f^q(V^j)$ leaves behind a subsurface L' of L that is f-invariant and has lost none of the interesting dynamics of f. Obviously no end of L' will be simple. If the closure of $\partial V^j \smallsetminus \partial L$ contains arcs, this procedure may introduce new noncompact boundary components, but it is evident that none of these can contain f-periodic points. The surviving ends that were periodic in L are periodic and cyclic in L' and no new cyclic ends have been introduced, hence $f : L' \to L'$ is endperiodic. Repeating this process for $j = 1, 2, \dots, r$ gives such a subsurface L' in which the end e survives as an end of type $< k$. Proceeding in this way, we finally arrive at an f-invariant subsurface $L' \subset L$ of type 0, without simple ends, and such that $f : L' \to L'$ is endperiodic. Note that, by Lemma 2.13, L' has only finitely many noncompact boundary components, each joining a positive end to a negative end.

Definition 2.14. The f-invariant subsurface $L' \subset L$ will be called the *soul* of L.

Of course, the soul is not unique – it can generally be enlarged or diminished and, in paring off nonperiodic ends, one may leave behind a lot of compact boundary components or many fewer.

2.2. **Finite Depth Foliations of Class C^2.** Suppose that $\mathcal{F}$ is a C^2, transversely oriented foliation of codimension one and finite depth > 0 of a compact, connected n-manifold M. We are going to need the Poincaré-Bendixson theory of finite depth foliations of class C^2 [6] [4, Section 8.4]. In these references, it was assumed that the leaves had empty boundary and were orientable. We wish to avoid these assumptions, so we give a short account. To begin with, it will be unnecessary to assume that $n = 3$.

Let F be a leaf at depth $k-1$ which is approached on a given side by a leaf L at depth k. In the given references, it is proven that there is an "octopus" decomposition $F = B \cup A_1 \cup A_2 \cup \cdots \cup A_r$ into connected submanifolds, where B is compact and is called the "nucleus", the "arms" A_i are disjoint, and $B \cap A_i = S_i$ is a compact component of ∂B. The proofs do not require that $\partial F = \emptyset$ nor that F be orientable. Similarly, the proof that the holonomy contraction on F, induced by the asymptotic leaf L, defines a cohomology class $\eta \in H^1_c(F;\mathbb{Z})$, compactly supported in $B \smallsetminus S_1 \smallsetminus \cdots \smallsetminus S_r$, goes through with no difficulty. The compactness of this class is due to the C^2 hypothesis. One next wants a compact, codimension 1, transversely oriented submanifold $N \subset B \smallsetminus S_1 \smallsetminus \cdots \smallsetminus S_r$ such that the intersection number of a loop $\sigma \subset F$ with N is equal to $\eta(\sigma)$. Poincaré duality was used for this, but without orientability this is problematic. When $\partial F = \emptyset$ one shows that N can be chosen to be connected, but we will sacrifice this property. In the references, N is referred to as the "juncture" and, in fact, it lifts to infinitely many copies of itself in L which are junctures for an end in the sense that we have defined above.

We prove the following without assuming orientability.

Lemma 2.15. *There is a canonical map $D : H^1_c(F;\mathbb{Z}) \to H_{n-2}(F, \partial F;\mathbb{Z})$ which, if F is oriented, is the Poincaré duality isomorphism. In any case, the class $D(\alpha)$, $\alpha \in H^1_c(F;\mathbb{Z})$, is represented by a compact, properly imbedded, transversely oriented submanifold N of codimension one such that, if $\sigma : S^1 \to L$ is in general position*

relative to N, the intersection number $\sigma \cdot N$, defined via the transverse orientation of N, is equal to $\alpha(\sigma)$.

Proof. Let $\alpha \in H^1_c(F;\mathbb{Z})$. As is well known, there is a map $f : F \to S^1$, unique up to homotopy, such that $f^*[S^1] = \alpha$, $[S^1]$ being the fundamental class of S^1. We may take f to be smooth and constantly $= q$ outside of a compact subset of F. By Sard's theorem, there is a regular value $p \neq q$ both for f and for $f|\partial F$. Then, $N = f^{-1}(p)$ is a properly imbedded submanifold of codimension one, transversely oriented by the orientation of S^1. Since $f \equiv q$ outside a compact set, N must be compact. Let $\sigma : S^1 \to F$ be in general position relative to N. Then $f \circ \sigma : S^1 \to S^1$ has p as a regular value and has degree equal to the number of times (counted with sign) that it crosses p. Clearly, this degree is equal to $\sigma \cdot N$ and is the value of α on σ. We set $D(\alpha) = [N]$ and remark that, in the case that F is orientable, $D(\alpha)$ is the Poincaré dual of α. □

We now consider the case that $\dim M = 3$, hence the leaves are 2-dimensional and $\dim N = 1$. The compactly supported cohomology class η is the one defined by the holonomy action of $\pi_1(F)$ on the intersection points of the leaf L with a transverse arc out of F. The ends of L asymptotic to F wind in on F in such a way that certain lifts of N to these ends form the system of junctures already described for periodic ends. This is the "infinite repetition" picture described in the above references.

Since the depth is positive, Reeb stability prevents the presence of leaves that are spheres, projective planes or disks. We suppose that ∂M has convex corners separating it into the subsurface $\partial_\tau M$ (the tangential boundary), the components of which are leaves of $\mathcal{F}$, and the subsurface $\partial_\pitchfork M$ transverse to $\mathcal{F}$ (the transverse boundary).

The Poincaré-Bendixson theory of finite depth foliations of class C^2 [6] [4, Section 8.4] actually implies that an isolated end of any leaf L at any depth is asymptotic only to a compact leaf F, winding in on that leaf as an "infinite repetition". Thus, if we assume that no compact leaf is a torus, a Klein bottle, a Möbius strip or an annulus, no leaf has a simple end. We fix this assumption.

A leaf of $\mathcal{F}$ is said to be *locally stable* if it admits a saturated neighborhood foliated as a product. Let $O \subset M$ be the union of locally stable leaves, an open, saturated set. Remark that the leaves of highest depth all lie in O.

If W is a component of O, L a leaf of $\mathcal{F}|W$, then L is of some topological type $k \geq 0$ [6, Theorem 6.0] and W is fibered over S^1 or $[0,1]$ with L as fiber. We ignore the latter case as uninteresting. Via a choice of smooth, oriented, one dimensional foliation $\mathcal{L}$ on M, tangent to $\partial_\pitchfork M$ and transverse to $\mathcal{F}$, one defines an associated monodromy diffeomorphism $f : L \to L$. Endperiodic maps of surfaces of topological type k are of interest, partly because of the following proposition.

Proposition 2.16. *The monodromy f of L is an endperiodic map and the soul L' of L has no simple ends.*

Proof. Let $\widehat{W}$ be the transverse completion of W, a manifold with convex corners and finitely many tangential boundary components (cf. [4, Section 5.2]). These tangential components of $\partial\widehat{W}$ are leaves of $\mathcal{F}$, although two such components might be identical as leaves of $\mathcal{F}$. There is an "octopus decomposition" of $\widehat{W}$ into a compact, connected "nucleus" H and finitely many "arms" $B_1, B_2, \dots, B_q$. The

nucleus H Is a manifold with boundary and corners. Parts of the boundary lie in leaves of $\mathcal{F}$ and constitute $\partial_\tau H$. The transverse boundary $\partial_\pitchfork H$ itself falls into two parts: the part that lies in $\partial_\pitchfork M$ and the part interfacing the arms, these being separated by corners. Each arm meets H only in a single annulus $\subseteq \partial_\pitchfork H$ or in a family of rectangles $\subset \partial_\pitchfork H$. The arms are fibered by subarcs of leaves of $\mathcal{L}$ and, if H is chosen large enough, the foliation $\mathcal{F}|B_i$ is a product foliation, $1 \le i \le q$. This is a consequence of the generalized Kopell lemma [4, Lemma 8.1.24 and Subsection 8.1.D] and makes essential use of the C^2 hypothesis. Finally, since no leaf has a simple end, one chooses H larger, if necessary, to insure that no component of $\partial_\tau H$ is an annulus, a Möbius strip or a disk. Of course, no component is a torus or Klein bottle. The leaves of $\mathcal{F}|B_i$ that are components of $L \cap B_i$ are neighborhoods of ends of L that are neither cyclic nor periodic, $1 \le i \le q$. Because of the triviality of $\mathcal{F}|B_i$, $L' = L \cap H$ is connected, being a leaf of $\mathcal{F}|H$. This is a depth one foliation of H and, since $\partial_\tau H$ contains no annulus, Möbius strip, torus or Klein bottle, the finitely many ends of L' are not simple. As it is well known that the monodromy of depth one foliations is endperiodic, the assertion follows, L' being the soul of L. □

Remark. Our general requirement that boundary components of L' connect positive ends to negative ends will be met if no component of $\partial_\pitchfork H$ is a Reeb-foliated annulus.

A principal goal in this paper is to classify endperiodic maps up to isotopy. The monodromy of a locally stable leaf depends on the choice of transverse, oriented foliation $\mathcal{L}$, and so for our theory to be useful for classifying the monodromy of L, the following elementary lemma is needed.

Lemma 2.17. *If W is a component of O, L a leaf of $\mathcal{F}|O$, and*

$$f_0, f_1 : L \to L$$

two monodromy maps defined by different choices of smooth, transverse, one-dimensional foliations, oriented by the transverse orientation of L, then f_0 and f_1 are smoothly isotopic through endperiodic diffeomorphisms f_t, $0 \le t \le 1$.

Proof. Let $\mathcal{L}_i$ be the transverse foliation determining f_i, $i = 0, 1$. Let v_i be a smooth, nonsingular vector field, tangent to $\mathcal{L}_i$ and agreeing with the transverse orientation of $\mathcal{F}$. Then $v_t = tv_1 + (1-t)v_0$, $0 \le t \le 1$, defines a homotopy of these fields through smooth, nonsingular vector fields transverse to $\mathcal{F}$. These define a smooth, nonsingular vector field v on $M \times I$, tangent to the factors $M \times \{t\}$ and to the leaves of a smooth, one-dimensional foliation $\mathcal{L}$ that restricts to a foliation $\mathcal{L}_t$ on $M \times \{t\}$ transverse to $\mathcal{F} \times \{t\}$. The first return map induced by $\mathcal{L}$ on $L \times I$ is a smooth homotopy of f_0 to f_1 through the endperiodic monodromy diffeomorphisms f_t on $L \times \{t\}$ induced by $\mathcal{L}_t$. □

2.3. **Finite depth foliations of Class C^0.** We now assume that the foliation $\mathcal{F}$, otherwise as above, is only integral to a C^0 plane field. Such a foliation admits a smooth, transverse, one-dimensional foliation $\mathcal{L}$ and each leaf of $\mathcal{F}$ has a C^∞ structure inherited from the transverse C^∞ structure of $\mathcal{L}$. If $\partial M = \emptyset$, a theorem of S. E. Goodman [21] [4, Theorem 6.3.5] and the fact that no leaf is a torus or a Klein bottle implies that the foliation is taut. (For the case in which M is nonorientable, one applies Goodman's argument in the 2-fold orientation cover and

then projects back into M.) This theorem is usually proven in the smooth case so that the Euler class of the tangent bundle of $\mathcal{F}$ can be defined by differential forms. But the class can also be defined using singular cohomology and essentially the same proof goes through. Since we allow both $\partial_\tau M \neq \emptyset$ and $\partial_\pitchfork M \neq \emptyset$, it is not immediately obvious that Goodman's theorem applies. Since we will need tautness, we reduce to Goodman's theorem as follows.

Lemma 2.18. *If no compact leaf of $\mathcal{F}$ is a torus, a Klein bottle, an annulus or a Möbius strip and if $\mathcal{F}|\partial_\pitchfork M$ has no two-dimensional Reeb components, then $\mathcal{F}$ is taut.*

Proof. By lifting to the 2-fold orientation cover, we can assume that M is orientable with no compact leaf a torus or annulus. Let $(M', \mathcal{F}')$ be formed by doubling along $\partial_\tau M$ and let $(M'', \mathcal{F}'')$ be formed by doubling along $\partial_\pitchfork M'$.

Since no leaf of $\mathcal{F}$ is a torus or annulus, we see that no leaf of $\mathcal{F}''$ is a torus. By Goodman's theorem, $\mathcal{F}''$ is taut. We will prove that this implies that $\mathcal{F}'$ is taut and that this, in turn, implies that $\mathcal{F}$ is taut.

If $\mathcal{F}'$ is not taut, some compact leaf F does not meet a closed transversal. Since $\mathcal{F}|\partial_\pitchfork M$, hence $\mathcal{F}'|\partial_\pitchfork M'$, has no Reeb components, any leaf meeting $\partial_\pitchfork M'$ meets a closed transversal, hence $F \subset \operatorname{int} M'$. Thus F survives as a leaf of $\mathcal{F}''$, but meets a closed transversal σ to that foliation. The closed transversal σ falls into segments $\sigma_1, \dots, \sigma_r$ lying alternately in M' and in the mirror image of M' in M''. Reflecting these second set of segments in $\partial_\pitchfork M'$, one obtains a loop in M' with corners. Since the reflection preserves the transverse orientation of $\mathcal{F}''$, these corners can be rounded to give an $\mathcal{F}'$-transverse loop in M' that meets F. This contradiction proves that $\mathcal{F}'$ is taut.

If $\mathcal{F}$ is not taut, some (necessarily closed) leaf F meets neither a closed transversal nor a transverse arc joining two components of $\partial_\tau M$. In M', F meets a closed $\mathcal{F}'$-transverse loop, some segment of which must issue from a component of $\partial_\tau M$, cross F, and terminate at another component of $\partial_\tau M$. This contradiction completes the proof. □

We will assume hereafter that no leaf is a torus, Klein bottle, annulus or Möbius strip and that $\mathcal{F}|\partial_\pitchfork M$ has no Reeb components. In the C^2 case, this easily ruled out simple ends, but this is not obvious here as it is no longer true that isolated ends are asymptotic only to compact leaves. Nevertheless, we prove that, under our hypotheses, no leaf of $\mathcal{F}$ has a simple end.

Recall that, if e is an end of a leaf L, its *asymptote* or *limit set* $A_e = \lim_e L$ [4, Definition 4.3.4] is a union of leaves of $\mathcal{F}$. If the highest depth of any leaf in A_e is $k-1$, then we will say that e is an end at depth k. Evidently, any leaf at depth k has at least one end at depth k.

Lemma 2.19. *Under our hypotheses, every neighborhood of every end of every noncompact leaf has fundamental group that is free on infinitely many generators. Consequently, the fundamental group of every noncompact leaf is also free on infinitely many generators and no leaf has a simple end.*

Proof. We proceed by induction on the depth k of the end e. If $k = 1$, the asymptote A_e reduces to a compact leaf F and each neighborhood of e contains a neighborhood that winds in on F as an "infinite repetition" [4, Definition 8.4.2] exactly as in the C^2 case. This is because the compactness of F implies the compactness of junctures.

Since F is neither a torus nor an annulus, the conclusion for the neighborhoods of e is immediate. In particular, the desired conclusion for leaves at depth one holds.

Inductively, for $k \geq 2$, assume that the assertion holds for all ends and all leaves at depth $k-1$. Let e be an end at depth k of a leaf L' and let $L \subset A_e$ be a leaf at depth $k-1$. Let $x_* \in L$ and let J be a subarc of a leaf of $\mathcal{L}$ having x_* as an endpoint and such that a neighborhood U of e in L' meets J infinitely often. For definiteness, assume that J is positively oriented out of x_* by the transverse orientation of $\mathcal{F}$ and note that $U \cap J = \{x_n\}_{n=0}^{\infty}$ where $x_n \downarrow x_*$ in J. By taking J short enough, we can guarantee that there is a path $\sigma \subset U$ joining x_0 to x_1 and staying in a normal neighborhood of L with normal fibers subarcs of leaves of $\mathcal{L}$. Projection along the normal fibers carries σ to a loop s in L based at x_* whose holonomy carries x_n to x_{n+1}, $n \geq 0$. Let d be a loop in L, based at x_0 such that the elements $[s], [d] \in \pi_1(L, x_*)$ do not commute. Shortening J, if necessary, we can assume that the holonomy h_d of d is defined at every x_n and that the lifts of d to these points stay in U. Changing the orientation of d, if necessary, we can assume that h_d is nonincreasing on $U \cap J$. Then $h_d(x_n) = x_{n+q}$ for some fixed integer $q \geq 0$ and for all $n \geq 0$. The loops $s_n = s^n d s^{-n-q}$ define a set of free generators of a free subgroup of the free group $\pi_1(L, x_*)$ that lift to loops σ_n in U based at x_0. The loop σ_n is freely homotopic in M to s_n via projections along the normal fibers. Since the foliation is taut, L is π_1-injective and it follows that $\{\sigma_n\}_{n=0}^{\infty}$ are free generators of a subgroup of $\pi_1(M, x_0)$, hence of $\pi_1(U, x_0)$. Since U is an open surface, it follows that $\pi_1(U)$ is free on some (possibly finite) number ≥ 2 of generators. But this is true for *every* neighborhood of e in L', implying that $\pi_1(U, x_0)$ is free on infinitely many generators. All assertions follow. $\square$

For these foliations, it remains true that the union O of the locally stable leaves is an open, saturated set. For each component W of O, the octopus decomposition of $\widehat{W}$ also holds. The proofs of these facts are strictly C^0. If H is a nucleus of $\widehat{W}$, $\partial_\tau H$ contains no tori or Klein bottles and, by Lemma 2.19, H can be chosen large enough that $\partial_\tau H$ contains no annuli, Möbius strips or disk leaves either. Thus, the depth one foliation $\mathcal{F}|H$ has all leaves with no simple ends and the Handel-Miller theory that we treat in this paper will apply to these leaves. It is no longer true that the foliation in the arms is a product foliation, and so the leaves L of $\mathcal{F}|W$ themselves do not have endperiodic monodromy, but each component of $L \cap H$ does and this will enable us to extend the theory of [10] to locally stable leaves of higher depth in taut, finite depth foliations integral to C^0 plane fields [5]. These are precisely the types of finite depth foliations constructed by Gabai [16, 18, 19].

2.4. **Foliation Cones.** In [10], we gave a cohomological classification, up to isotopy, of depth one foliations. This made essential use of the Handel-Miller theory. Assuming that all compact leaves are components of $\partial_\tau M$, we found a finite family of convex, non-overlapping, polyhedral cones in $H^1(M) = H^1(M; \mathbb{R})$ such that the isotopy classes of depth one foliations corresponded exactly to the rays through integer lattice points in the interiors of these cones. In [5], we will extend this to the components W of O, classifying all stable refoliations of W. This will use our adaptation of Handel-Miller theory to these higher depth cases. For C^2 foliations, we use the generally infinite dimensional vector space $H^1_\kappa(\widehat{W}) \subseteq H^1(\widehat{W})$ consisting of the classes represented by compactly supported cocycles. Again there will be a

finite family of convex, nonoverlapping, polyhedral cones classifying these refoliations up to isotopy. The cones are generally infinite dimensional, but have finitely many faces. (This results from the fact, proven in Section 9, that the "core dynamical system" of the Handel-Miller monodromy is Markov.) For C^0 foliations, the completely analogous classification produces such cones in $H^1(\widehat{W})$. The discussion in Subsections 2.2 and 2.3 sets the stage for this classification.

The lamination $\mathcal{F}|(M \smallsetminus O)$ can be called the "substructure" of $\mathcal{F}$. The theory proposed above will then amount to the classification of all finite depth foliations with a given substructure.

3. Hyperbolic Structures and Limit Sets

Let L be a complete, connected hyperbolic surface with geodesic boundary and without simple ends. It may be orientable or nonorientable.

Definition 3.1. A hyperbolic surface L with the above properties will be called *admissible.*

It is not excluded that L might be compact, although our primary interest is in the noncompact case.

What we assemble here are largely facts that are well known or for which proofs can easily be supplied by readers reasonably well versed in hyperbolic geometry. Therefore, we omit some details.

One can form the hyperbolic surface $2L$ by doubling along the geodesic boundary of L. The condition that L have no simple end is equivalent to the condition that $2L$ have no isolated planar end. Many results are more easily proven for surfaces without boundary. One often proves a result for L by proving the corresponding result for $2L$.

The foliations considered in Subsection 2.2 admit leafwise hyperbolic metrics in which all leaves are admissible. The condition that no end be simple can be reformulated geometrically.

Definition 3.2. A *half-plane* is a hyperbolic surface isometric to the union of a geodesic $\gamma \subset \Delta$ (the Poincaré disk) and one of the components of $\Delta \smallsetminus \gamma$.

Lemma 3.3. *The hyperbolic surface L has no simple end if and only if $2L$ has no imbedded half-planes and no imbedded cusps.*

Indeed, an isolated planar end in a complete hyperbolic surface without boundary is either a cusp or a "flaring" end. In the second case, there is an imbedded half-plane. This, in fact, is the only way an imbedded half-plane can occur. Remark that the absence of cusps implies that the covering transformations of $2L$, hence of L, are hyperbolic.

The absence of simple ends readily leads to the following.

Lemma 3.4. *If L is admissible, every end of $2L$ has a fundamental system of closed neighborhoods V_k such that ∂V_k is a simple closed geodesic.*

Corollary 3.5. *If L is admissible and orientable, there is a family of disjoint, simple closed geodesics that partitions $2L$ into pairs of pants.*

Let $\widetilde{L}$ be the universal cover of L and let $\overline{L}$ be the closure of $\widetilde{L}$ in the closed Poincaré disk $\Delta \cup S^1_\infty$ (where the circle at infinity S^1_∞ is the unit circle and the

Poincaré disk Δ is the open unit disk). Either $\partial L = \emptyset$, in which case $\widetilde{L} = \Delta$ and $\overline{L} = \Delta \cup S^1_\infty$, or $\widetilde{L}$ has geodesic boundary in Δ and $\overline{L}$ is the union of $\widetilde{L}$ and a compact set in S^1_∞. We will denote $\Delta \cup S^1_\infty$ by $\mathbb{D}^2$.

Definition 3.6. The *limit points* of L are the accumulation points in S^1_∞ of the set $\{\gamma(x_0) \mid \gamma \text{ a deck transformation of } \widetilde{L}\}$ for fixed $x_0 \in \widetilde{L}$. The union Y of these points is the limit set of L.

The following is well known and elementary.

Lemma 3.7. *The limit set of L is independent of $x_0 \in \widetilde{L}$.*

Let $X \subset S^1_\infty$ be the set of fixed points of the (extensions to $\overline{L}$ of the) deck transformations of $\widetilde{L}$. Let $E = \overline{L} \cap S^1_\infty$. Then $X \subset Y \subseteq E$. The following is a consequence of the fact that $2L$ has no imbedded half-planes.

Theorem 3.8. *For the admissible surface L, X is dense in E and $Y = E$.*

Corollary 3.9. *If $x \in E$, then the orbit of x under the group of covering transformations is dense in E.*

Corollary 3.10. *If $\widetilde{L} \neq \Delta$, then E is a Cantor set.*

Our next goal is to prove the following.

Theorem 3.11. *Suppose that L is an admissible surface and $h : L \to L$ is a homeomorphism. Then any lift $\widetilde{h} : \widetilde{L} \to \widetilde{L}$ extends canonically to a homeomorphism $\widehat{h} : \overline{L} \to \overline{L}$.*

Remark. We prove the theorem for $2L$ and the doubled homeomorphism $2h$. The truth of the theorem for L follows trivially. Hereafter, then, $\partial L = \emptyset$, $\widetilde{L} = \mathbb{D}^2$ and $E = S^1_\infty$. Also, as the orientation cover of nonorientable L is intermediate between L and the universal cover, we can assume without loss of generality that L is orientable.

For γ a geodesic in L, let $\overline{\gamma} \subset \mathbb{D}^2$ denote completion of a lift $\widetilde{\gamma} \subset \Delta$ by the endpoints on S^1_∞.

Lemma 3.12. *If $\sigma \subset L$ is an essential closed curve, then $\overline{\sigma} \subset \mathbb{D}^2$ is defined, for each lift $\widetilde{\sigma}$.*

This is well known. Tighten σ to a closed geodesic γ and complete $\widetilde{\sigma}$ by the endpoints of $\overline{\gamma}$.

Lemma 3.13. *There is a bijection $\widehat{h} : \mathbb{D}^2 \to \mathbb{D}^2$ which extends $\widetilde{h}$, carries each completed geodesic $\overline{\gamma}$, $\gamma \subset L$ closed, to a continuous curve in $\mathbb{D}^2$ with endpoints on S^1_∞, and is a homeomorphism on S^1_∞.*

Proof. We only need to define $\widehat{h} : S^1_\infty \to S^1_\infty$ appropriately. Since each orbit in S^1_∞ of (the extension to $\mathbb{D}^2$ of) the covering group is dense (Corollary 3.9), the set A of points x in S^1_∞ that are endpoints of all completed lifts $\overline{\gamma}$, where $\gamma \subset L$ ranges over the closed geodesics, is dense in S^1_∞. By Lemma 3.12, $\widetilde{h}$ naturally induces $\widehat{\widetilde{h}} : A \to A$, which we will denote by $\widehat{h}$. If $\widetilde{h}$ is orientation preserving, $\widehat{h}$ preserves the cyclic order in the dense set A, hence extends to a homeomorphism (again denoted by $\widehat{h}$) with the desired property. If $\widetilde{h}$ is orientation reversing, apply the

above reasoning to $\psi \circ \widetilde{h}$ where ψ is an orientation reversing isometry of Δ. Since $\widehat{\psi \circ \widetilde{h}} : S^1_\infty \to S^1_\infty$ is a homeomorphism, so is $\widehat{h} := \psi^{-1} \circ \widehat{\psi \circ \widetilde{h}}$. $\square$

Set $\mathcal{G}$ equal to the set of all completed lifts $\overline{\gamma}$ of all closed geodesics $\gamma \subset L$. For $x \in S^1_\infty$, let $\mathcal{G}_x \subset \mathcal{G}$ be the subset of those $\overline{\gamma}$ that do not have x as an endpoint.

Definition 3.14. For $x \in S^1_\infty$ and $\overline{\gamma} \in \mathcal{G}_x$, let $H_{\overline{\gamma}}$ be the closed half-plane in $\mathbb{D}^2$ with $\overline{\gamma}$ as boundary such that $x \in H_{\overline{\gamma}}$.

Remark. The closed half-plane $H_{\overline{\gamma}}$ is convex in the strong sense that any two of its points are joined by an arc of a unique completed geodesic $\overline{\lambda}$. The intersection of sets that are strongly convex in this sense is strongly convex. Thus, $C_x = \bigcap_{\overline{\gamma} \in \mathcal{G}_x} H_{\overline{\gamma}}$ is strongly convex.

Lemma 3.15. *No completed geodesic arc $\sigma \subset C_x$ with x as an endpoint can properly cross any completed lift $\overline{\gamma}$ of any closed geodesic γ.*

Proof. If $\overline{\gamma} \in \mathcal{G}_x$, this would contradict the fact that $\sigma \subset C_x$. Otherwise, $\overline{\gamma}$ and σ share the endpoint x and either $(\sigma \smallsetminus \{x\}) \cap \overline{\gamma} = \emptyset$ or $\sigma \subset \overline{\gamma}$. $\square$

Lemma 3.16. *There is a locally finite family of simple closed geodesics partitioning L into geodesic polygons.*

This follows from Corollary 3.5, which gives a preliminary pair of pants decomposition via disjoint closed geodesics. Since there are no isolated planar ends, there is a lot of topology around, making it possible to find a locally finite family of simple closed geodesics which cross every boundary geodesic of every pair of pants in the decomposition. Details are left to the reader.

Lemma 3.17. $C_x = \{x\}$.

Proof. Suppose there exists $y \in C_x$ distinct from x. C_x is strongly convex, hence the complete geodesic arc $[x, y]$ lies in C_x. Without loss of generality, we can choose y to be a finite point. Then $\sigma = \pi(x, y] \subset L$ is either a half-infinite geodesic or a closed geodesic, where $\pi : \Delta \to L$ is the covering projection. This cannot be contained in any of the polygons of Lemma 3.16, hence σ properly crosses a closed geodesic, contradicting Lemma 3.15. $\square$

Lemma 3.18. *For $x \in S^1_\infty$ and U an open, Euclidean neighborhood of x in $\mathbb{D}^2$, there exist $\overline{\gamma}_1, \dots, \overline{\gamma}_n \in \mathcal{G}_x$ such that $\bigcap_{i=1}^n H_{\overline{\gamma}_i} \subset U$.*

Proof. Let $B = \mathbb{D}^2 \smallsetminus U$, a compact set. By Lemma 3.17, $\bigcap_{\overline{\gamma} \in \mathcal{G}_x} (H_{\overline{\gamma}} \cap B) = \emptyset$. By the finite intersection property of compact sets it follows that there exists $\overline{\gamma}_1, \dots, \overline{\gamma}_n \in \mathcal{G}_x$ such that $\bigcap_{i=1}^n (H_{\overline{\gamma}_i} \cap B) = \emptyset$. Thus $V = \bigcap_{i=1}^n H_{\overline{\gamma}_i} \subset U$. Clearly V is a closed neighborhood of x. $\square$

In the following proof, H_i denotes the $H_{\overline{\gamma}_i}$ in the above lemma.

Proof of Theorem 3.11. We want to show that $\widehat{h} : \mathbb{D}^2 \to \mathbb{D}^2$ is continuous at x. This is clear if $x \in \Delta$, so we assume $x \in S^1_\infty$. Let U be an open neighborhood of $\widehat{h}(x)$. By Lemma 3.18, $U \supset V$ where V is a neighborhood of $\widehat{h}(x)$ of the form $V = \bigcap_{i=1}^n H_i$. Let $W = \widehat{h}^{-1}(V) = \bigcap_{i=1}^n \widehat{h}^{-1}(H_i)$. Now H_i subtends an arc $\alpha_i \subset S^1_\infty$ with $\widehat{h}(x) \in \operatorname{int} \alpha_i$ and so, by Lemma 3.13, $\widehat{h}^{-1}(\alpha_i)$ is an arc having x in its interior. Furthermore, again by Lemma 3.13, $\widehat{h}^{-1}(\overline{\gamma}_i)$ is a curve β_i in $\mathbb{D}^2$ with the same

endpoints as $\widehat{h}^{-1}(\alpha_i)$ and meeting S^1_∞ exactly in these endpoints. Thus $\widehat{h}^{-1}(H_i)$ is exactly the closed region in $\mathbb{D}^2$ bounded by $\alpha_i \cup \beta_i$ and so is a closed neighborhood of x. Thus $\widehat{h}^{-1}(V)$ is a closed neighborhood of x contained in $\widehat{h}^{-1}(U)$. We have proven that $\widehat{h} : \mathbb{D}^2 \to \mathbb{D}^2$ is continuous. hence is a homeomorphism. □

Remark. Our proof includes the case in which L is compact, hence gives a fundamentally different proof in that case from the ones given by Casson and Bleiler [2, Lemma 3.7] and Handel and Thurston [23, Corollary 1.2]. These proofs make use of compactness, whereas we only need the pair of pants decomposition. The analogous result holds for higher dimensional, compact, hyperbolic manifolds [1, Proposition C.1.2], [25, Theorem 11.6.2], where compactness is only used to guarantee that $\widetilde{h}$ is a pseudo-isometry. In [2], compactness is only used to guarantee that $\widetilde{h}$ is uniformly continuous.

4. The Laminations

We assume that $f : L \to L$ is endperiodic and that L has type 0, satisfying the hypotheses in Section 3. It can be assumed that f is an isometry in the fundamental neighborhoods of the ends and that the components of junctures are geodesics of length ≤ 1. We can also assume that the compact components of ∂L are closed geodesics of length 1.

Consider a juncture J_e near a negative end e of L and all positive and negative iterations of it under f. If the components of J_e are simple closed curves, each of these iterates is freely homotopic to a unique disjoint union J^n_e of simple closed geodesics. If J_e is a union of arcs, each arc of each iteration $f^n(J_e)$ is homotopic, with endpoints fixed, to a geodesic arc, the disjoint union of which we denote by J^e_n. These assertions are well known. Letting e range over all negative ends and n over all integers, we obtain geodesic 1-manifolds J^n_e that we will call the negative h-junctures (in anticipation of the Handel-Miller homeomorphism h that we are going to construct). These include the "honest" junctures in fundamental neighborhoods of negative ends. Similarly, we define the positive h-junctures. For an h-juncture J, we will denote by J^k the h-juncture corresponding to $f^k(J)$, $-\infty < k < \infty$.

Positive h-junctures remain of bounded length in positive directions and negative h-junctures remain of bounded length in negative directions. In the opposite directions, the lengths become unbounded as we will see.

Lemma 4.1. *If J is an honest juncture for an end e and if n_e is the integer introduced on* page 3*, then J^k and J^{k+n_e} cobound a compact subsurface diffeomorphic to a fundamental domain for e, $-\infty < k < \infty$.*

Proof. We can take J to be imbedded deeply enough in a fundamental neighborhood of e so that $f^{n_e}(J)$ is also a juncture in a fundamental neighborhood of that end. The 1-manifolds $f^k(J)$ and $f^{k+n_e}(J)$ cobound the compact surface $f^k(B)$, where B is the fundamental domain cobounded by J and $f^{n_e}(J)$, hence the geodesic 1-manifolds J^k and J^{k+n_e} cobound a compact subsurface diffeomorphic to B. □

Our definitions have allowed a trivial sort of endperiodic homeomorphism which we need to exclude, namely a "total translation". This is the case if L has one positive end and one negative end and if the fundamental domains of these ends form a bi-infinite sequence $\{B_i\}_{i=-\infty}^{\infty}$ filling up L. Thus, with suitable choice of

indexing, $f^{n_e}(B_i) = B_{i+1}$, $-\infty < i < \infty$. We give a useful characterization of this case.

Lemma 4.2. *The endperiodic map is a total translation iff, for each compact subset $K \subset L$, all but finitely many of the h-junctures lie in $L \smallsetminus K$.*

Proof. The condition is clearly necessary. Suppose the condition holds and that $J = J_e$ is an honest juncture for a negative end e. Thus, there is a fundamental neighborhood U_0 of a positive end e' and an integer $m \geq 0$ such that $J^m \cap U_0 = (J^m)' \neq \emptyset$ for $(J^m)'$ a suitable union of components of J^m. Since J, hence J^m, separates L into two components, U_0 and $(J^m)'$ can be chosen so that $(J^m)'$ separates U_0, and hence separates L into two components. It follows that $(J^m)' = J^m$. Thus, all h-junctures $J^{m+qn_{e'}}$ lie in U_0, where $q \geq 0$. By Lemma 4.1, partition a neighborhood of e' into a sequence $B_m \cup B_{m+1} \cup \cdots \cup B_{m+r} \cup \cdots$ of nonoverlapping submanifolds, each diffeomorphic to a fundamental domain B for e. Of course, a fundamental neighborhood of e is also so partitioned and another application of Lemma 4.1 allows us to join these two neighborhoods by a finite sequence of such copies of B meeting along suitable J^r's. Actually, one of these copies might chance to overlap B_m, but a different choice of the integer m will fix this. This gives a 2-ended subsurface which is clearly all of L and we can homotope the h-junctures J^r back to $f^r(J)$, concluding that f is a total translation. □

From now on, we exclude the case of a total translation. We do note, however, that the reduction process of Section 8 may produce pieces on which f is a total translation. This presents no problem since they are merely pieces that do not meet our laminations.

4.1. **Construction of the Laminations.** We will say that a subset $A \subset L$ *f-escapes* if, for every compact subset $K \subset L$, all but finitely many $f^k(A)$ lie in $L \smallsetminus K$, $-\infty < k < \infty$.

Lemma 4.3. *Endpoints of arc components of junctures f-escape.*

Proof. Since noncompact boundary components have no periodic points, any of their points f-escapes. Since any circle boundary component of L lying in a fundamental neighborhood of an end f-escapes in one direction, it must f-escape in the other, because a sequence of distinct closed boundary components cannot accumulate anywhere in L. Our assertion follows. □

If e is a positive end, let $\{U_i^e\}_{i=0}^{\infty}$ be a fundamental neighborhood system and, for each $i \geq 0$, denote by W_i^+ the union of the U_i^e's as e ranges over the positive ends. Similarly define W_i^-, for each $i \geq 0$.

Lemma 4.4. *For each $i \geq 0$ and each choice J of negative h-juncture, all but finitely many components of $\bigcup_{n\geq 0} J^n$ meet W_i^+ and, similarly, if J is a positive h-juncture, all but finitely many components of $\bigcup_{n\geq 0} J^{-n}$ meet W_i^-.*

Proof. If the negative h-juncture J is a union of arc components, the assertion about its subsequent h-junctures is immediate by Lemma 4.3. If J is a union of simple closed curves, we suppose there is a positive integer N such that $J^n \cap W_i^+ = \emptyset$, $n \geq N$. If N is large enough, we also have that $J^n \cap W_i^- = \emptyset$. It follows from Lemma 4.1 that the compact submanifold $K_i = L \smallsetminus (W_i^+ \cup W_i^-)$ contains infinitely many nonoverlapping compact submanifolds of negative Euler characteristic, which is clearly absurd. □

Definition 4.5. The compact submanifold K_i in the above proof will be called the i^{th} core of L. We will write $K = K_0$ and call it simply the core.

We are assuming that f is not a total translation, hence no negative h-juncture J^n can lie in W_0^+. But examples show that a component σ_n of J^n can lie in W_0^+. It follows that, if σ_k is the corresponding component of J^k, the sequence $\{\sigma_k\}_{k=0}^{\infty}$ clusters on a set of positive ends. As $k \downarrow -\infty$, the sequence obviously converges on a set of negative ends. By analogy with terminology introduced above and anticipating the definition of the Handel-Miller map in Subsection 4.2, we will say that σ_n "h-escapes". There is a similar discussion for positive h-junctures. Remark that, if σ_n f-escapes, it h-escapes, but the converse is false.

Corollary 4.6. *If J is a negative h-juncture and σ is a component of J, then either σ h-escapes, or the lengths of σ_k become unbounded as $k \uparrow \infty$. The corresponding assertion holds for components of positive h-junctures.*

Proof. If σ does not h-escape, each σ_k, $k \geq 0$, must have a point $x_k \in L \smallsetminus W_0^+$ and, by Lemma 4.4, another point $y_k \in W_{i_k}^+$, where i_k becomes arbitrarily large as $k \uparrow \infty$. Thus, the distance between x_k and y_k becomes unbounded and the assertion follows. □

Definition 4.7. Let $\mathcal{X}_-$ (respectively, $\mathcal{X}_+$) denote the union of negative (respectively, positive) h-junctures. Define $\Gamma_\pm = \overline{\mathcal{X}}_\mp$ and set $\Lambda_\pm = \Gamma_\pm \smallsetminus \mathcal{X}_\mp$.

Corollary 4.8. *Every component σ of $\mathcal{X}_\pm$ either h-escapes or meets infinitely many components of $\mathcal{X}_\mp$.*

Proof. Say σ is a component of $\mathcal{X}_-$ that does not h-escape and let σ_n be the component of $\mathcal{X}_-$ homotopic to $f^n(\sigma)$. By Lemma 4.4, the number of positive junctures that σ_n crosses becomes unbounded as $n \uparrow \infty$. Hence, applying f^{-n} and performing the usual homotopies establishes the claim for $\mathcal{X}_-$. The proof for $\mathcal{X}_+$ is entirely similar. □

Lemma 4.9. *If the geodesic component σ of $\mathcal{X}_-$ does not h-escape, then the σ_n's contain points x_n such that x_n converges to a positive end as $n \uparrow \infty$ and points y_n that stay in a bounded region of L. The corresponding assertion holds for geodesic components of $\mathcal{X}_+$.*

Proof. For an h-juncture J in a neighborhood of a positive end, Lemma 4.2 and Lemma 4.4 imply that, for $n \geq 0$ large enough, a component τ_{-n} of J^{-n} meets a nonescaping component σ of a negative juncture. Applying f^n to these intersecting curves gives intersecting curves $f^n(\tau_{-n})$ and $f^n(\sigma)$. Now the first of these is homotopic to a geodesic component τ of J and the geodesic σ_n, homotopic to the second, must intersect τ in a point x_n. Since the juncture J can be chosen arbitrarily close to the positive end, this procedure leads to the desired sequences. If there are not also points $y_n \in \sigma_n$ remaining in a bounded region of L, then σ must h-escape. □

Definition 4.10. A complete geodesic $\lambda \subset L$ is a locally uniform limit of a family $\mathcal{Z}$ of geodesics if, for every compact subarc $[a,b] \subset \lambda$ there is a normal neighborhood $[a,b] \times (-\varepsilon, \varepsilon)$ such that, whenever $0 < \delta < \varepsilon$, all but finitely many components of $\mathcal{Z} \cap ([a,b] \times (-\delta, \delta))$ are transverse to the normal fibers and have one endpoint in $\{a\} \times (-\delta, \delta)$ and the other in $\{b\} \times (-\delta, \delta)$.

Remark that, in the above definition, we allow that $a = b$ and that $[a, b] = \lambda$ is a closed geodesic. In this case, $\{a\} \times (-\varepsilon, \varepsilon)$ is identified with $\{b\} \times (-\varepsilon, \varepsilon)$, possibly with an orientation reversing twist. Thus, the normal neighborhoods are either annuli or Möbius strips.

Proposition 4.11. *The closed subsets $\Lambda_\pm$ are laminations of L, the leaves of which are complete, one-to-one immersed geodesics without endpoints, each a locally uniform limit of geodesics in $\mathfrak{X}_\mp$, and $\Lambda_\pm \cap \mathfrak{X}_\mp = \emptyset$. The lamination Λ_+ does not meet a fundamental neighborhood of any negative end and Λ_- does not meet such a neighborhood of any positive end.*

Proof. Let $x \in \Lambda_+$. Then there is a sequence $\{\sigma_n\}_{n=1}^\infty$ of components of negative h-junctures and points $x_n \in \sigma_n$ such that $x_n \to x$ as $n \uparrow \infty$. Lifting to the universal cover $\widetilde{L}$, pick $\widetilde{x}$ covering x and $\widetilde{x}_n$ covering x_n, $1 \le n < \infty$, such that $\widetilde{x}_n \to \widetilde{x}$. Let $\widetilde{x}_n \in \widetilde{\sigma}_n$, where $\widetilde{\sigma}_n$ covers σ_n. If infinitely many of the σ_n's are closed geodesics, we can pass to a subsequence so that all are. Thus $\widetilde{\sigma}_n$ is a complete geodesic in $\widetilde{L}$, hence a circular arc with distinct endpoints $\{w_n, z_n\} \subset S^1_\infty$. Again passing to a subsequence, we can assume that $w_n \to w$ and $z_n \to z$, both points in S^1_∞. Since $\widetilde{x}$ does not lie on the circle at infinity, $w \neq z$. The complete geodesic $\widetilde{\sigma}$ in $\widetilde{L}$ with endpoints w, z is the uniform limit of $\{\widetilde{\sigma}_n\}_{n=1}^\infty$ (in the Euclidean topology of $\overline{L} \subset \mathbb{D}^2$) and contains $\widetilde{x}$, hence the projection $\sigma \subset L$ is a complete geodesic through x without endpoints, lies in Λ_+ and is the locally uniform limit of $\{\sigma_n\}_{n=1}^\infty$. Now suppose that $\widetilde{\sigma}_n$ is a geodesic arc joining points $\{w_n, z_n\} \subset \partial\widetilde{L}$. If a subsequence of w_n's converges to a point $w \notin S^1_\infty$, we would get a contradiction to Lemma 4.3, and similarly for the z_n's. The proof now proceeds as above.

Thus, $\Lambda_\pm$ is the union of such geodesics and, if two of them properly intersected or if one properly intersected itself or if one intersected an element of $\mathfrak{X}_\mp$, two elements of $\mathfrak{X}_\mp$ would properly intersect, which is clearly impossible.

Finally, elements of $\mathfrak{X}_-$ cannot accumulate in a fundamental neighborhood of a negative end, nor can $\mathfrak{X}_+$ accumulate in such a neighborhood of a positive end, proving the final assertion. □

Proposition 4.12. *No leaf of $\Lambda_\pm$ lies entirely in a bounded region of L.*

Proof. Suppose that the leaf λ does lie in a bounded region. For definiteness, assume $\lambda \in \Lambda_-$. Choose the i^{th} core K_i so that $\lambda \subset \operatorname{int} K_i$. Let $x \in \lambda$ and choose a sequence $\{J^n\}_{n=1}^\infty \subset \mathfrak{X}_+$ and $x_n \in J^n$ such that $x_n \to x$ as $n \to \infty$. For n large, $J^n \not\subset K_i$ (Lemma 4.9) and we let $I^n \subset J^n$ be the unique component of $J^n \cap K_i$ containing x_n. The geodesic arc I^n has endpoints a_n and b_n in negative junctures in ∂K_i. Any two of these arcs are disjoint since the elements of $\mathfrak{X}_+$ are disjoint and have no proper self intersections. Passing to a subsequence, we can assume that $a_n \to a$ and $b_n \to b$ monotonically in ∂K_i. Thus, the geodesic arcs I^n converge uniformly to a geodesic arc $I \subset K_i$ through x with endpoints a and b. By Definition 4.7 , I lies in a leaf of Λ_-, hence in λ. But this contradicts the assumption that $\lambda \subset \operatorname{int} K_i$. □

Definition 4.13. An end ε of a curve s in L passes arbitrarily near an end e of L if every fundamental neighborhood of e meets every neighborhood of ε. In this case we also say that s passes arbitrarily near e.

Notice that a curve can pass arbitrarily near e and still return repeatedly to the core $K \subset L$. In fact, in the geodesic laminations that we have constructed, this is the typical behavior of the leaves.

Corollary 4.14. *Each end of every leaf of Λ_+ (respectively, Λ_-) passes arbitrarily near a positive (respectively, negative) end of L, but does not meet a fundamental neighborhood of a negative (respectively, positive) end. Furthermore, if e is an end of L, every neighborhood of e meets $\Lambda_\pm$.*

Proof. Consider the case of a leaf $\lambda_+ \subset \Lambda_+$, not compact by Proposition 4.12. If one of its ends ε does not pass arbitrarily near any positive end of L, the asymptote of ε is a nonempty, compact sublamination of Λ_+. This contradicts Proposition 4.12. The second assertion is by Proposition 4.11. The final assertion is an easy consequence of Corollary 4.8. □

Proposition 4.15. *The laminations $\Lambda_\pm$ are mutually transverse and each leaf of one meets a leaf of the other.*

Proof. Suppose a leaf of Λ_+ meets a leaf of Λ_-. If the intersection is not transverse, the two geodesics coincide, giving a common leaf ℓ of both laminations. By the above, such a leaf both does and does not meet a fundamental neighborhood of some end, hence the intersection is transverse.

Finally, let λ_+ be a leaf of Λ_+ and let $\{\sigma_n\}_{n=1}^\infty$ be a sequence of components of negative h-junctures converging locally uniformly to λ_+. We can assume that all $\sigma_n \subset L \smallsetminus W_0^-$. For each n, σ_n meets infinitely many components of $\mathfrak{X}_+$ (Corollary 4.8), all but finitely many of which must be of the form $\tau_n \subset L \smallsetminus W_0^+$. Let $x_n \in \sigma_n \cap \tau_n$. These are points in the compact submanifold $K = L \smallsetminus (W_0^+ \cup W_0^-)$ and so, passing to a subsequence, we may assume that $x_n \to x \in K$. Since the σ_n's converge locally uniformly to λ_+, we know that $x \in \lambda_+$. But, as above, a subsequence of the τ_n's converges locally uniformly to a leaf λ_- of Λ_- and so $x \in \lambda_-$ also. □

Remark that $\Lambda_+ \cap \Lambda_-$ is a compact, totally disconnected subset of the core K.

Since there was some freedom in the choice of junctures and $\mathfrak{X}_\pm$ depends on the choices, one is reasonably concerned that the laminations $\Lambda_\pm$ might also depend on these choices. The following relieves this concern.

Proposition 4.16. *The laminations $\Lambda_\pm$ are independent of the choice of junctures.*

Proof. Let Λ_+ result from one choice of negative junctures and Λ'_+ from another. Call the two sets of h-junctures $\mathfrak{X}_-$ and $\mathfrak{X}'_-$, respectively. Let ℓ be a leaf of Λ_+, a locally uniform limit of a sequence of negative h-junctures in $\mathfrak{X}_-$. Since there are only finitely many repelling ends, we can pass to a subsequence of the form $\{J_{k_i}\}_{i=1}^\infty$ clustering locally uniformly on ℓ, where $J = J^0 \subset \mathfrak{X}_-$ is a juncture for a negative end e. Furthermore, for any fixed choice of $r > 0$, The sequence $\{J^{k_i+rn_e}\}_{i=1}^\infty$ also clusters locally uniformly on ℓ, where n_e is the positive integer in Lemma 4.1. We choose $r > 0$ so that a juncture $J' = (J^0)' \subset \mathfrak{X}'_-$ lies in the compact surface cobounded by J^0 and J^{rn_e}. Thus $(J^{k_i})'$ also clusters uniformly on ℓ as $i \to \infty$. Since ℓ was chosen arbitrarily, we have that $\Lambda_+ \subset \Lambda'_+$. The reverse inclusion is proven in exactly the same way, as is the equality $\Lambda_- = \Lambda'_-$. □

The sets $\Gamma_\pm$ are laminations of L. Some leaves are closed geodesics and some are properly imbedded geodesic arcs. These laminations do, of course, depend on the choice of junctures.

Definition 4.17. $\Gamma_\pm$ are called the *extended laminations*.

Remark. If $\partial L \neq \emptyset$, we double to obtain a surface $2L$ without boundary. Those junctures that were unions of geodesic arcs become "junctures" that are unions of closed, piecewise geodesic circles. The proof of Proposition 4.16 shows that these accumulate on exactly the same laminations as the corresponding geodesic junctures. The laminations in $2L$ are the union of two copies of the laminations in L. A number of proofs are more easily carried out for surfaces without boundary and this remark shows that no real generality is lost by restricting to that case.

4.2. **Construction of the Homeomorphism h on $\Gamma_\pm \cup \partial L$.** Our ultimate goal is to define an endperiodic map $h : L \to L$, isotopic to f through endperiodic maps, which preserves the above laminations. To begin with, we will define h on $\Lambda_\pm$, $\mathcal{X}_\pm$ and ∂L. After analyzing the properties of this map we will extend it to all of L.

Consider those compact components σ_0 of ∂L for which there is a least integer $n \geq 1$ with $f^n(\sigma_0) = \sigma_0$, and set $\sigma_i = f^i(\sigma_0)$, $0 \leq i \leq n$. Here $\sigma_n = \sigma_0$. Fix choices of $x_i \in \sigma_i$, $0 \leq i \leq n$, with $x_0 = x_n$. Since these boundary components are closed, oriented geodesics of length 1, There is a unique orientation preserving isometry $h : \sigma_i \to \sigma_{i+1}$ such that $h(x_i) = x_{i+1}$, $0 \leq i \leq n-1$. This defines h on these boundary components so that $h^n = \mathrm{id}$ on each of them. On all other components of ∂L, compact or not, we set $h = f$.

Let $\widetilde{f} : \widetilde{L} \to \widetilde{L}$ be a lift of $f : L \to L$ to the universal cover. Let G denote the group of covering transformations. If $\psi \in G$, there is a unique $\psi' \in G$ such that $\psi' \circ \widetilde{f} = \widetilde{f} \circ \psi$. The map $\widetilde{f}$ extends continuously and canonically to a homeomorphism $\widehat{f} : \overline{L} \to \overline{L}$ (Theorem 3.11). This trivially yields the following.

Lemma 4.18. *If $\widetilde{\mathcal{G}}$ is the set of geodesics in $\widetilde{L}$ with both endpoints on E or on $\partial\widetilde{L}$, then $\widetilde{f}$ induces a canonical map, again called $\widehat{f} : \widetilde{\mathcal{G}} \to \widetilde{\mathcal{G}}$. Covering transformations also act on $\widetilde{\mathcal{G}}$ and again we have $\psi' \circ \widehat{f} = \widehat{f} \circ \psi$, $\forall \psi \in G$.*

The geodesic laminations $\Gamma_\pm$ lift to geodesic laminations $\widetilde{\Gamma}_\pm$ of $\widetilde{L}$, as do the laminations $\Lambda_\pm$.

If $\{x\} = \widetilde{\gamma}_+ \cap \widetilde{\gamma}_-$ with $\widetilde{\gamma}_\pm$ leaves of $\widetilde{\Gamma}_\pm$, define $\widetilde{h}(x) = \widehat{f}(\widetilde{\gamma}_+) \cap \widehat{f}(\widetilde{\gamma}_-)$. Since $\widehat{f}$ is a homeomorphism on E, we obtain a homeomorphism $\widetilde{h} : \widetilde{\Gamma}_+ \cap \widetilde{\Gamma}_- \to \widetilde{\Gamma}_+ \cap \widetilde{\Gamma}_-$. If ψ is a covering transformation, then $\widetilde{h}(\psi(x)) = \psi'(\widetilde{h}(x))$ by Lemma 4.18. Consequently,

Lemma 4.19. *The homeomorphism $\widetilde{h} : \widetilde{\Gamma}_+ \cap \widetilde{\Gamma}_- \to \widetilde{\Gamma}_+ \cap \widetilde{\Gamma}_-$ descends to a well defined homeomorphism $h : \Gamma_+ \cap \Gamma_- \to \Gamma_+ \cap \Gamma_-$ as well as $h : \Lambda_+ \cap \Lambda_- \to \Lambda_+ \cap \Lambda_-$*

We next extend $\widetilde{h}$ linearly over each geodesic segment in $\widetilde{\Gamma}_- \cup \widetilde{\Gamma}_+ \smallsetminus (\Gamma_+ \cap \Gamma_-)$, obtaining a homeomorphism (again denoted by $\widetilde{h}$) of this set to itself. Since covering transformations are isometries, $\widetilde{h}$ descends to a well-defined homeomorphism

$$h : \Gamma_- \cup \Gamma_+ \to \Gamma_- \cup \Gamma_+.$$

If we define $h|\partial L = f|\partial L$, this is compatible with the above since endpoints of h-junctures agree with the endpoints of the corresponding f-junctures. This gives a homeomorphism

$$h : \Gamma_- \cup \Gamma_+ \cup \partial L \to \Gamma_- \cup \Gamma_+ \cup \partial L.$$

Finally, we show that the homeomorphism $h : \Gamma_+ \cup \Gamma_- \cup \partial L \to \Gamma_+ \cup \Gamma_- \cup \partial L$ defines uniquely a homeomorphism $\widehat{h} : E \to E$ that coincides with $\widehat{f}$. The key lemma is the following.

Corollary 4.20. *The endpoints of $\widetilde{\mathfrak{X}}_\pm$ and of $\widetilde{\Lambda}_\pm$ each constitute dense subsets of E.*

Indeed, each of these sets is invariant under the group of covering transformations, hence the claim is immediate by Corollary 3.9. (Indeed, since $\widetilde{\Lambda}_\pm$ is closed, the set of its endpoints is all of E.) Since $\widehat{f}$ and $\widehat{h}$ are both defined and equal on these subsets of E, we get the promised conclusion:

Corollary 4.21. *The homeomorphism $h : \Gamma_+ \cup \Gamma_- \cup \partial L \to \Gamma_+ \cup \Gamma_- \cup \partial L$ induces uniquely a homeomorphism $\widehat{h} : E \to E$ that coincides with $\widehat{f}$.*

4.3. **Relaxing the Geodesic Condition.** The use of hyperbolic geodesics was useful for constructing the laminations, but in applications and in constructing examples it is useful to relax the condition that the laminations and junctures be geodesic. We give here a preliminary set of axioms for these more general Handel-Miller systems associated to the endperiodic map f and, in all the subsequent discussion, we will prove theorems only on the basis of these axioms. These axioms have already been proven for the geodesic laminations. After proving extensive structure theorems via these axioms, we will be able to prove, in the geodesic case, that $h : \Gamma_+ \cup \Gamma_- \cup \partial L \to \Gamma_+ \cup \Gamma_- \cup \partial L$ extends to $h : L \to L$. We will then replace the preliminary axioms with a partially new set and use the final axiom system to prove the "transfer theorem" 11.1.

Definition 4.22. A topologically immersed line or circle in L will be called a pseudo-geodesic if an arbitrary lift to the universal cover $\widetilde{L}$ has distinct, well-defined endpoints on the circle at infinity. A properly immersed arc will also be called a pseudo-geodesic.

Of course, geodesics are pseudo-geodesics. A closed curve is a pseudo-geodesic if and only if it is essential. We emphasize that these curves are only continuous, hence possibly wild. Normally, we will only consider one-to-one immersed or imbedded pseudo-geodesics.

Axiom 1. Λ_+ and Λ_- are mutually transverse, pseudo-geodesic laminations with all leaves noncompact and disjoint from ∂L. Furthermore, the leaves of the lifted laminations $\widetilde{\Lambda}_\pm$ are determined by their endpoints on S^1_∞.

Remark. We understand that laminations, even very nonsmooth ones, are to be *locally trivial.* That is, the lamination is covered by topological product neighborhoods $I \times I$ in L such that each leaf intersects such a neighborhood in a collection of plaques $\{t_\alpha \times I\}_{\alpha \in \mathfrak{A}}$.

Axiom 2. The laminations Λ_+ and Λ_- are *strongly closed* in the following sense:

Definition 4.23. The lamination $\Lambda_\pm$ is strongly closed if it is closed and, whenever a sequence x_n in $\widetilde{\Lambda}_\pm$ converges to $x \in \widetilde{L}$, the leaves $\ell_n \subset \widetilde{\Lambda}_\pm$ through x_n converge uniformly in the *Euclidean* metric on Δ to a leaf $\ell \subset \widetilde{\Lambda}_\pm$ through x (in the sense that there are arbitrarily small Euclidean normal neighborhoods N of ℓ in Δ such that all but finitely many ℓ_n lie in N and are transverse to the normal fibers).

Furthermore, whenever the endpoints of a sequence of leaves $\ell_n \subset \widetilde{\Lambda}_\pm$ converge to a pair of points $x, y \in E \subset S^1_\infty$, then these are the endpoints of a leaf $\ell \subset \widetilde{\Lambda}_\pm$ and $\ell_n \to \ell$ uniformly in the Euclidean metric.

Corollary 4.24. *The laminations $\Lambda_\pm$ are bounded away from each compact component of ∂L*

Otherwise, Axiom 2 would imply that some leaf meets a compact component of ∂L, contradicting Axiom 1.

The next two axioms ensure that Λ_+ and Λ_- are related to each other just like the Handel-Miller geodesic laminations.

Axiom 3. Every leaf of $\widetilde{\Lambda}_\pm$ meets at least one leaf of $\widetilde{\Lambda}_\mp$ and can do so only in a single point.

The next axiom is Corollary 4.14 for the geodesic case.

Axiom 4. Each end of every leaf of Λ_+ (respectively, of Λ_-) passes arbitrarily near a positive (respectively, negative) end of L, but does not enter the fundamental neighborhoods of negative (respectively, positive) ends. Furthermore, if e is an end of L, every neighborhood of e meets $\Lambda_\pm$.

We next relate the laminations to the endperiodic map $f : L \to L$. There will be mutually transverse families $\mathcal{X}_\pm$ of disjoint, essential, compact (hence, pseudo-geodesic), properly imbedded 1-manifolds which will play the role of "h-junctures". The family $\mathcal{X}_\pm$ is to be disjoint from the lamination $\Lambda_\mp$. It will be assumed that, for each positive (respectively, negative) end e of L, there is a juncture $J_e = J_e^0$ in a fundamental neighborhood of e such the the elements of J_e^n of $\mathcal{X}_+$ (respectively, of $\mathcal{X}_-$) correspond one-to-one to the 1-manifolds $f^n(J_e)$, as e ranges over the positive (respectively, negative) ends of L and n ranges over $\mathbb{Z}$. The correspondence is isotopy, with endpoints (if any) fixed.

The junctures J_e and $J_e^{n_e}$ (see page 3) cobound a fundamental domain $B_e = B_e^0$. We further assume the following which, in the geodesic case, is Corollary 4.1.

Axiom 5. J_e^n and $J_e^{n+n_e}$ cobound a compact manifold B_e^n homeomorphic to B_e.

Definition 4.25. The compact manifold B_e^n is called a positive (respectively, negative) h-fundamental domain if e is a positive (respectively, negative) end.

Lemma 4.26. *If e and e' are both negative (respectively, positive) ends and the h-fundamental domains B_e^n and $B_{e'}^m$ have nonempty intersection, then $e = e'$ and either $n = m$, or $|n - m| = 1$ and they intersect along a common negative (respectively, positive) juncture.*

Proof. For definiteness, suppose that e and e' are negative ends. Then B_e^n and $B_{e'}^m$ are closures of components of $\mathcal{U}_- \smallsetminus \mathcal{X}_-$. Distinct ones can only intersect in a subset of their boundaries. Since distinct junctures in $\mathcal{X}_-$ are disjoint and B_e^n is completely determined by the two junctures that constitute its boundary, the assertion is evident. □

Definition 4.27. We set $\mathcal{U}_e = \bigcup_{n=-\infty}^{\infty} B_n^e$. The union of the sets $\mathcal{U}_e$ as e ranges over the negative (respectively, positive) ends will be denoted by $\mathcal{U}_-$ (respectively, $\mathcal{U}_+$). We will call $\mathcal{U}_-$ the negative escaping set and $\mathcal{U}_+$ the positive escaping set.

The set $\mathcal{U}_e$ is clearly open and connected.

Remark. When, ultimately, we extend the homeomorphism h over L, the positive (respectively, negative) escaping set will be exactly the positive (respectively, negative) h-escaping set.

Remark that, in general, $\mathcal{U}_- \cap \mathcal{U}_+ \neq \emptyset$.

Corollary 4.28. *If $e \neq e'$ are both positive or both negative ends, then $\mathcal{U}_e \cap \mathcal{U}_{e'} = \emptyset$.*

By Corollary 4.28, the $\mathcal{U}_e$'s are the connected components of $\mathcal{U}_-$, as e ranges over the negative ends, and the connected components of $\mathcal{U}_+$, as e ranges over the positive ends.

Remark. It should be obvious that $\mathcal{X}_\pm$ is a family of curves that are isolated from each other.

Definition 4.29. $\mathcal{X}_+$ is the family of positive h-junctures and $\mathcal{X}_-$ the family of negative ones.

In what follows we blur the distinction between the family $\mathcal{X}_\pm$ and the union of its elements.

Axiom 6. $\overline{\mathcal{X}}_\mp \smallsetminus \mathcal{X}_\mp = \Lambda_\pm$ and the curves in $\mathcal{X}_\mp$ accumulate locally uniformly on the leaves of $\Lambda_\pm$ (where we adapt Definition 4.10 to pseudo-geodesics).

Corollary 4.30. Axiom 6 *holds in the geodesic case.*

This is by construction and Proposition 4.11.

We set $\Gamma_\pm = \overline{\mathcal{X}}_\mp$. This is a closed lamination called the "extended lamination".

Lemma 4.31. *The frontier of $\mathcal{U}_\mp$ is $\Lambda_\pm$.*

Proof. Let x be a point of the frontier of $\mathcal{U}_\mp$. One can construct a curve s joining a point $y \in \mathcal{U}_\mp$ to x. Since $x \notin \mathcal{U}_\mp$, it cannot lie in any h-fundamental domain. Thus the interior of s must meet $\mathcal{X}_\mp$ and these points must cluster at x. These points can be taken in distinct components of $\mathcal{X}_\mp$, hence they cannot accumulate in that set. Thus $x \in \Lambda_\pm$ by Axiom 6. For the reverse inclusion, let $x \in \Lambda_\pm$ and choose a sequence $x_n \in \mathcal{X}_\mp$ converging to x (Axiom 6). Since each x_n is a boundary point of an h-fundamental domain, x must be an accumulation point of $\mathcal{U}_\mp$. By Axiom 6, $\Lambda_\pm \cap \mathcal{X}_\mp = \emptyset$, hence $\Lambda_\pm \cap \mathcal{U}_\mp = \emptyset$, and so x must lie in the frontier of $\mathcal{U}_\mp$. □

Axiom 7. The lifted laminations $\widetilde{\Gamma}_\pm$ and $\widetilde{\mathcal{X}}_\pm$ in $\widetilde{L}$ are transverse and a curve in one can only meet a curve in the other in a single point.

This axiom holds in the geodesic case by construction and the fact that the curves are hyperbolic geodesics.

Axiom 8. There is a homeomorphism

$$h : \Gamma_+ \cup \Gamma_- \to \Gamma_+ \cup \Gamma_-$$

which induces a homeomorphism $\widehat{h}$ on E that coincides with $\widehat{f}$, $\widehat{h}$ being defined exactly as in Subsection 4.2.

For the geodesic case, this axiom was proven in Subsection 4.2. Remark that the equality $\widehat{h} = \widehat{f}$ implies that h permutes the h-junctures J_e^n exactly as f permutes the corresponding f-junctures $f^n(J_e)$. For this reason we may think of h as "endperiodic", taking care to remember, however, that it is not yet defined on all of L.

Definition 4.32. Given an endperiodic map $f : L \to L$, the triple (Γ_+, Γ_-, h) satisfying Axioms 1 to 8 will be called a Handel-Miller system associated to f.

Our construction via geodesic laminations has established the existence of a Handel-Miller system associated to f.

Remark. We can extend h to an automorphism of $\Gamma_+ \cup \Gamma_- \cup \partial L$ by taking $h|\partial L = f|\partial L$. We can also take the double of L, doubling the endperiodic map f, the laminations $\Lambda_\pm$ and the h-junctures. Since the points of ∂L escape to positive and negative ends under forward and backward iterations of f, one quickly checks that the axioms remain true in this setting. Thus, **we lose no generality in restricting our attention to the case of empty boundary and we will do so in the remainder of the paper, save notice to the contrary**. Thus, all components of h-junctures will be simple closed curves and $\widetilde{L} = \Delta$. The family of boundary curves of L survive as an h-invariant family $\mathcal{Z}_\partial$ which will be part of the h-invariant family $\mathcal{Z}$ of *reducing curves* that we will be building as we go.

The following proposition, which is stated for the general case, will illustrate both the convenience of working in the case of empty boundary and the fact that no generality is lost.

Proposition 4.33. *Let λ be a leaf of $\Lambda_\pm$, $x \in \lambda$, and let $\{x_n\}_{n=1}^\infty \subset \mathcal{X}_\mp$ converge to x. Let σ_n be the component of $\mathcal{X}_\mp$ through x_n, $1 \le n < \infty$, and let $\widetilde{\lambda}, \widetilde{x}, \widetilde{x}_n, \widetilde{\sigma}_n$ be lifts to $\widetilde{L}$ with $\widetilde{x} \in \widetilde{\lambda}$, $\widetilde{x}_n \in \widetilde{\sigma}_n$ and $\widetilde{x}_n \to \widetilde{x}$. Then the endpoints of $\widetilde{\sigma}_n$ converge to the endpoints of $\widetilde{\lambda}$ in the Euclidean metric of $\overline{\Delta}$.*

Proof. To begin with, assume that $\partial L = \emptyset$. Since $\widetilde{\Lambda}_\pm$ is a lamination, we can also assume that the convergence $\widetilde{x}_n \to \widetilde{x}$ is strictly monotone along an arc transverse to $\widetilde{\Lambda}_\pm$. Since $\widetilde{\sigma}_n \cap \widetilde{\sigma}_m = \emptyset$, $m \neq n$, we can choose the endpoints $a, b \in S^1_\infty$ of $\widetilde{\lambda}$ and $a_n, b_n \in S^1_\infty$ of $\widetilde{\sigma}_n$ so that $a < a_{n+1} < a_n$ in the counterclockwise sense and $b < b_{n+1} < b_n$ in the clockwise sense. If $a_n \to a' \neq a$, then some component $\widetilde{\sigma} \subset \widetilde{\mathcal{X}}_\mp$ has an endpoint in the counterclockwise arc in S^1_∞ from a' to a. This is because of the fact that $\widetilde{\mathcal{X}}_\mp$, being invariant under the covering group, has endpoints dense in S^1_∞ (Corollary 3.9). But then $\widetilde{\sigma}$, being disjoint from every $\widetilde{\sigma}_n$, is trapped between $\widetilde{\lambda}$ and all $\widetilde{\sigma}_n$, contradicting the fact that $\widetilde{x}_n \to \widetilde{x}$.

Assume now that $\partial L \neq \emptyset$ and double, obtaining $2L = L \cup F$, where F is a copy of L. The universal cover of $2L$ is Δ, formed from countably many copies of $\widetilde{L}$ and $\widetilde{F}$ in the usual way. Let $\mathcal{L}_\pm \supset \Lambda_\pm$ be the laminations of $2L$. If the σ_n's are all circles, the above argument works, hence we assume that they are all properly imbedded arcs. But the doubles $\tau_n = 2\sigma_n$ are circles and h-junctures in $2L$. Suppose that the endpoints a_n, b_n of the $\widetilde{\sigma}_n$'s do not converge to the endpoints of the leaf $\widetilde{\lambda} \subset \widetilde{\Lambda}_\pm$. By the previous paragraph, the endpoints of $\widetilde{\tau}_n$ do converge to the endpoints of $\widetilde{\lambda}$ and, passing to a subsequence, we can assume that $a_n \to a' \neq a$. By Axiom 6, this places a leaf $\widetilde{\ell} \neq \widetilde{\lambda}$ of $\widetilde{\mathcal{L}}_\pm$ between $\widetilde{\lambda}$ and all $\widetilde{\tau}_n$'s. This leaf would have to have the same endpoints as $\widetilde{\lambda}$, contradicting Axiom 1. □

Continuing with the situation in this proposition, let $\overline{\lambda} \subset \overline{\Delta}$ be the completion of $\widetilde{\lambda}$ by the addition of its endpoints and define $\overline{\sigma}_n$ similarly. Then Axiom 6 and Proposition 4.33 imply the following.

Corollary 4.34. *For each Euclidean normal neighborhood N of $\overline{\lambda}$ in $\overline{\Delta}$, $\overline{\sigma}_n \subset N$ for all but finitely many values of n.*

Since the accumulation of $\mathcal{X}_\mp$ on $\Lambda_\pm$ is only *locally* uniform, the σ_n's cannot be assumed to be transverse to all of the normal fibers of N. They may develop "kinks" arbitrarily far out near S^1_∞ as $n \to \infty$.

In all that follows, we fix Axioms 1-8, dispensing with the requirement that the laminations be geodesic.

4.4. **Internal Completion and the Border of Open Sets.** Let $U \subseteq L$ be an open, connected subset and define the "path metric" d as follows. Given two points x, y in this open, connected set, there are piecewise geodesic paths connecting these points. Each such path has a well-defined length and we define $d(x, y)$ to be the greatest lower bound of these lengths. It is standard that this defines a topological metric on U compatible with the topology of this open set. Notice that two points can be very far apart in the metric d and very close together in the hyperbolic metric of L.

Definition 4.35. Define the *internal completion* $\widehat{U}$ to be the completion of U in the metric d and define the *boundary* of $\widehat{U}$ to be $\partial\widehat{U} = \widehat{U} \smallsetminus U$.

As is standard, the metric d extends to a metric on $\widehat{U}$ which we will again denote by d.

While $\widehat{U}$ and $\partial\widehat{U}$ are not generally subsets of L, the inclusion map $\iota : U \hookrightarrow L$ is a topological imbedding. It extends canonically to a continuous map $\widehat{\iota} : \widehat{U} \to L$. This map may be very pathological on $\partial\widehat{U}$, but in the cases occurring in this paper, it will be an immersion on this set and often will be a one-to-one immersion on each of its components.

Definition 4.36. The image under $\widehat{\iota}$ of a component of $\partial\widehat{U}$ will be called a *border component* of U. The set δU of border components of U will be called the *border* of U.

Since, generally, two border components might intersect properly, we do not take δU to be their union.

These notions extend to open sets that are not connected. If U is such, $\widehat{U}$ denotes the disjoint union of the internal completions of each component. Correspondingly, $\partial\widehat{U}$ is defined separately for each component, $\widehat{\iota} : \widehat{U} \to L$ is defined componentwise, and δU is the set of images under $\widehat{\iota}$ of components of $\partial\widehat{U}$.

Lemma 4.37. *The boundary $\partial\widehat{U}$ can be characterized as the set of points $x \in \widehat{U}$ such that there exists a continuous map $s : [0, 1] \to \widehat{U}$ such that $s(1) = x$ and $s(t) \in U$, $0 \le t < 1$.*

Proof. Such a point clearly is in $\partial\widehat{U}$. Conversely, if $x \in \partial\widehat{U}$, there is a Cauchy sequence (relative to the metric d) $\{x_n\}_{n=1}^\infty \subset U$ that converges to x. Let $\varepsilon_n > d(x_n, x_{n+1})$ such that $\lim_{n\to\infty} \varepsilon_n = 0$. Then there is a piecewise geodesic path $s_n \subset U$ joining x_n to x_{n+1} of length $< \varepsilon_n$. Every point in this path is within ε_n of x_{n+1}. Joining these paths sequentially and suitably parametrizing produces a continuous map $s : [0, 1) \to U$ such that $\lim_{t\to 1} s(t) = x$. Thus, s is extended continuously to $[0, 1]$ so that $s(1) = x$. $\square$

Applying $\widehat{\iota}$ to the above picture gives the following.

Corollary 4.38. *If $x \in \ell \in \delta U$, there exists a continuous map $s : [0, 1] \to L$ such that $s(1) = x$ and $s(t) \in U$, $0 \le t < 1$.*

The sets U that we will be considering have $\partial\widehat{U}$ and δU made up piecewise of subarcs of $\Gamma_+ \cup \Gamma_-$. Each such subarc is the image of possibly two such subarcs in $\partial\widehat{U}$.

5. The Complementary Regions to the Laminations

We identify the components of $L \smallsetminus \Lambda_\pm$. They fall into two essentially different types: the escaping sets (positive and negative, cf. Definition 4.27) and the principal regions.

Recall that, in accordance with the remark on page 24, we are now assuming that $\partial L = \emptyset$.

5.1. **The Positive and Negative Escaping Sets.**

Recall that the laminations $\Lambda_\pm$, are the frontiers $\partial\mathcal{U}_\mp$ (Axiom 6).

Lemma 5.1. *If e is a negative (respectively, positive) end, then $\mathcal{U}_e$ is a component of $L \smallsetminus \Lambda_+$ (respectively, of $L \smallsetminus \Lambda_-$).*

Proof. Indeed, $\mathcal{U}_e$ is connected and lies in the complement of Λ_+. But $\partial\mathcal{U}_e \subseteq \partial\mathcal{U}_- = \Lambda_+$. $\square$

Proposition 5.2. *If $\lambda_+ \subset \Lambda_+$ (respectively $\lambda_- \subset \Lambda_-$) is a border leaf of a component $\mathcal{U}_e$ of $\mathcal{U}_-$ (respectively, of $\mathcal{U}_+$), then negative h-junctures (respectively, positive h-junctures) accumulate locally uniformly on λ_+ (respectively, on λ_-) on any side that borders $\mathcal{U}_e$.*

Proof. Suppose $\lambda_+ \subset \Lambda_+ \cap \delta\mathcal{U}_e$ where $\mathcal{U}_e$ is a component of $\mathcal{U}_-$. Let $x \in \lambda_+$, let V be an arbitrary neighborhood of x in $\widehat{\mathcal{U}}_e$ and let $y \in V \cap \mathcal{U}_e$. Then $y \in B^e_j$ for some j, $-\infty < j < \infty$. If only finitely many values of j occur, then λ_+ would meet one of these h-fundamental domains, a contradiction. Thus $V \cap J^e_j \neq \emptyset$ for infinitely many values of j. Since the neighborhood V of x is arbitrary and $x \in \lambda_+$ is arbitrary, the assertion follows by Axiom 6. The case in which λ_- is a border leaf of a component of $\mathcal{U}_+$ is handled similarly. $\square$

5.2. **Principal Regions.**

Definition 5.3. The set $\mathcal{P}_+$ is the union of the components of the complement of the lamination Λ_+ that contain no points in $\mathcal{X}_-$, i.e. no points of negative h-junctures. We define a *positive principal region* to be a component of $\mathcal{P}_+$. Similarly, the set $\mathcal{P}_-$ consists of the components of the complement of the lamination Λ_- that contain no points in $\mathcal{X}_+$, i.e. no points of positive h-junctures, and we define a *negative principal region* to be a component of $\mathcal{P}_-$.

Lemma 5.4. *$L = \mathcal{U}_- \cup \Lambda_+ \cup \mathcal{P}_+$ and $L = \mathcal{U}_+ \cup \Lambda_- \cup \mathcal{P}_-$ where the unions are disjoint.*

Proof. Suppose V is a component of $L \smallsetminus \Lambda_+$ that meets a negative h-juncture. Since some component $\mathcal{U}_e$ of $L \smallsetminus \Lambda_+$ contains that entire h-juncture, $V = \mathcal{U}_e$. $\square$

Definition 5.5. The *escaping set* is

$$\mathcal{U} = \mathcal{U}_- \cap \mathcal{U}_+ = (L \smallsetminus (\Lambda_+ \cup \mathcal{P}_+)) \cap (L \smallsetminus (\Lambda_- \cup \mathcal{P}_-)) = L \smallsetminus (\Lambda_- \cup \mathcal{P}_- \cup \Lambda_+ \cup \mathcal{P}_+).$$

When, in Section 8, we extend the homeomorphism h to all of L, $\mathcal{U}$ will be the set of points that escape to both positive and negative ends under iterations of $h^{\pm 1}$.

Definition 5.6. The *invariant set* is $\mathcal{I} = L \smallsetminus (\mathcal{U}_- \cup \mathcal{U}_+)$.

Of course, $\mathcal{I}$ will be the set of points that do not escape to any ends under any extension of h to L.

Lemma 5.7. $\mathcal{I} = (\Lambda_+ \cap \Lambda_-) \cup (\Lambda_+ \cap \mathcal{P}_-) \cup (\Lambda_- \cap \mathcal{P}_+) \cup (\mathcal{P}_+ \cap \mathcal{P}_-)$

Proof.

$$\begin{aligned} \mathcal{I} &= L \smallsetminus (\mathcal{U}_- \cup \mathcal{U}_+) \\ &= (L \smallsetminus \mathcal{U}_-) \cap (L \smallsetminus \mathcal{U}_+) \\ &= (\Lambda_+ \cup \mathcal{P}_+) \cap (\Lambda_- \cup \mathcal{P}_-) \\ &= (\Lambda_+ \cap \Lambda_-) \cup (\Lambda_+ \cap \mathcal{P}_-) \cup (\Lambda_- \cap \mathcal{P}_+) \cup (\mathcal{P}_+ \cap \mathcal{P}_-) \end{aligned}$$

□

Lemma 5.8. $\mathcal{P}_+ \subset \mathcal{I} \cup \mathcal{U}_+$ *and* $\mathcal{P}_- \subset \mathcal{I} \cup \mathcal{U}_-$.

Proof. If $x \in \mathcal{P}_+$ and $x \notin \mathcal{U}_+$, then $x \in L \smallsetminus (\mathcal{U}_- \cup \mathcal{U}_+) = \mathcal{I}$. Thus $\mathcal{P}_+ \subset \mathcal{I} \cup \mathcal{U}_+$. Similarly, $\mathcal{P}_- \subset \mathcal{I} \cup \mathcal{U}_-$. □

6. Semi-isolated Leaves

A leaf $\lambda_\pm$ of $\Lambda_\pm$ will either be *isolated* in $\Lambda_\pm$ (i.e. approached by points of $\Lambda_\pm$ on neither side) or approached by points of $\Lambda_\pm$ on one or both sides.

Definition 6.1. If a leaf $\lambda_\pm \subset \Lambda_\pm$ is approached by leaves of $\Lambda_\pm$ on at most one side, we say $\lambda_\pm$ is *semi-isolated*.

It should be clear that h, being continuous on $\Lambda_\pm$, carries semi-isolated leaves to semi-isolated leaves.

Note that our definition of semi-isolated includes all isolated leaves. It will turn out that there are only finitely many semi-isolated leaves and that each contains an h-periodic point. Our analysis will cast more light on $\mathcal{U}_\pm$ and $\mathcal{P}_\pm$. For all of this, we need some analysis of the core $K \subset L$.

6.1. The Core.

We fix fundamental neighborhood systems

$$U_0^e \supset U_1^e \supset \cdots \supset U_n^e \supset \cdots$$

for each attracting or repelling end e. Let W^+ denote the union of the U_0^e's as e ranges over the positive ends and let W^- be similarly defined for the negative ends. As already defined for the endperiodic map f, the compact, connected core $K = L \smallsetminus (W^+ \cup W^-)$ will also serve as the core for h. Remark that, by renumbering the fundamental neighborhood systems so that U_1^e is called U_0^e, etc., the core is effectively enlarged so as to guarantee that, if e is a negative end and $J^e \subseteq \delta K$ is the juncture separating off U^e from K, then $h^{n_e}(J^e) \subset K$. If e is a positive end, then $h^{-n_e}(J^e) \subset K$. This will be useful in several subsequent arguments.

Remark that $\chi(K) < 0$. Otherwise, K would be an annulus and our endperiodic map f would be a total translation.

Remark. The core is not unique, but depends on the choices of the U_0^e.

Lemma 6.2. *Each leaf of* $\Lambda_\pm$ *meets* $\operatorname{int} K$.

Proof. By Axiom 3, each leaf λ of $\Lambda_\pm$ meets some leaf λ' of $\Lambda_\mp$. Since λ cannot meet $\overline{W}^\mp \subset \mathcal{U}_\mp$ and λ' cannot meet $\overline{W}^\pm \subset \mathcal{U}_\pm$, we have $\emptyset \neq \lambda \cap \lambda' \subset \operatorname{int} K$. □

Definition 6.3. If C is the closure of a component of $K \smallsetminus \Lambda_{\pm}$, then a *side* of C is an arc of $C \cap \Lambda_{\pm}$.

Remark. Any component of the boundary of the closure C of a component of $K \smallsetminus \Lambda_{\pm}$ will either consist of a circle in ∂K or alternately of sides of C and arcs in ∂K.

By abuse of language, we will often use the term "component of $K \smallsetminus \Lambda_{\pm}$" both for that component and its closure.

Definition 6.4. If R is a component of $K \smallsetminus \Lambda_{-}$ with exactly two sides, both in Λ_{-}, and if these sides are isotopic in R, then R is called a *positive rectangle*. One defines *negative rectangles* analogously.

Note that positive rectangles are bounded by two arcs of Λ_{-} and two arcs of $\mathcal{X}_{-}$, and analogously for negative rectangles. A positive rectangle is either a component of $K \cap \mathcal{U}_{+}$ or of $K \cap \mathcal{P}_{-}$, with an analogous statement for negative rectangles. Thus, it is really a toss-up which to call positive and which negative. Since the sets $\mathcal{U}_{\pm}$ are never empty and $\mathcal{P}_{\pm}$ often are, we have made the above choice.

It is tempting to apply h^k to components (such as rectangles) of $L \smallsetminus \Gamma_{\pm}$, but h and its powers are only defined on $\Gamma_{\pm}$. Here and later we will use the following for simply connected components.

Lemma 6.5. *If $D \subset L$ is a disk with $\partial(D) \subset \Gamma_{+} \cup \Gamma_{-}$, then $h(\partial D)$ also bounds a disk in L. This disk will be denoted by $h_*(D)$.*

Proof. Recall that $\widetilde{h}$ is defined on $\widetilde{\Gamma}_{+} \cup \widetilde{\Gamma}_{-}$. Any connected lift $\widetilde{D}$ of D is a disk and $\widetilde{h}$ is defined on $\partial \widetilde{D}$, mapping it to a simple closed curve that necessarily bounds a disk. Thus, if π denotes the covering projection, $\pi \widetilde{h}(\partial \widetilde{D}) = h(\partial D)$, the left hand side of the equation being homotopically trivial and the right hand side being a simple closed curve. ☐

Lemma 6.6. *If $R \subset K$ is a positive rectangle, then $h_*(R) \subset \operatorname{int} K$. Likewise, if $R \subset K$ is a negative rectangle, $h_*^{-1}(R) \subset \operatorname{int} K$.*

Proof. Evidently, $h(\partial R) \subset \operatorname{int} K$ for a positive rectangle. ☐

Consider all arcs of $\Lambda_{\pm} \cap K$ and consider the isotopy classes of such arcs *where the isotopy is to be through arcs with endpoints on components of ∂K in $\mathcal{X}_{\pm}$*. The track of such an isotopy is a rectangle bounded by the two arcs of $\Lambda_{\pm} \cap K$ in question and two arcs of ∂K in components of $\mathcal{X}_{\pm}$. Because $\Lambda_{\pm}$ is a closed lamination, the track of the full isotopy class of an arc α of $\Lambda_{\pm} \cap K$ is such a rectangle R_α. Remark that the "rectangle" R_α may degenerate into a single arc.

Lemma 6.7. *The rectangle R_α meets $\Lambda_{\pm}$ only in arcs isotopic to α.*

Proof. The leaves of $\Lambda_{\pm}$ cannot properly intersect one another, so any arc β of $\Lambda_{\pm} \cap K$ that meets R_α must be contained in that rectangle. The ends of β cannot lie in the same edge of R_α because this would form a digon with an edge in a leaf $\Lambda_{\pm}$ and an edge in a component of $\mathcal{X}_{\pm}$, contradicting Axiom 7. Thus β is isotopic to α. ☐

Lemma 6.8. *The arcs α of $\Lambda_{\pm} \cap K$ fall into finitely many isotopy classes, determining finitely many disjoint rectangles R_α.*

Proof. The rectangles are disjoint by Lemma 6.7. No arc α of $\Lambda_\pm \cap K$ can separate K into components, one of which is a disk. Again, this is because Axiom 7 forbids such digons. Cutting K apart along α either produces a connected surface K' with $0 \geq \chi(K') = \chi(K)+1$, or two components K_1, K_2 with $0 \geq \chi(K_i) > \chi(K)$, $i = 1, 2$. Notice that the arcs in a different isotopy class survive and cannot cut off a disk in the new surface(s). If a component with zero Euler characteristic occurs, it is an annulus or Möbius strip and contains at most one of the isotopy classes. Take this component out of play and cut along another arc in a different isotopy class, continuing in a process that must terminate after finitely many steps. □

Proposition 6.9. *There are at most finitely many components of $K \smallsetminus \Lambda_\pm$ which are not rectangles and each such component has finitely many sides.*

Proof. By Lemma 6.8, each of the finitely many rectangles R_α either consists of a single arc or has two "extreme" arcs in $\Lambda_\pm \cap K$ and these finitely many arcs are the only candidates for sides of nonrectangular components. □

Definition 6.10. The non-rectangular components of $K \smallsetminus \Lambda_\pm$ will be called the *special* components and the finitely many sides of these special components will be called the special components of $\Lambda_\pm \cap K$.

6.2. **Periodic Leaves.** We consider leaves $\lambda_\pm \in \Lambda_\pm$ which are periodic under h. For definiteness, assume that $\lambda_- \in \Lambda_-$ is h-periodic of period p. If h^p interchanges ends of λ_-, replace p by $2p$. We will study this situation in the universal cover $\widetilde{L} = \Delta$. There, we fix a lift $\widetilde{\lambda}_-$ and choose the lift $\widetilde{h}$ (defined on $\widetilde{\Gamma}_\pm$) so that $\widetilde{h}^p(\widetilde{\lambda}_-) = \widetilde{\lambda}_-$, noting that $\widetilde{h}^p$ fixes the endpoints of this leaf on S^1_∞.

We get three possibilities by taking the closure $\overline{\sigma}$ of a lift $\widetilde{\sigma}$ of a component of $\mathfrak{X}_-$ which meets $\widetilde{\lambda}_-$ and iterating using $\widetilde{h}^p$. Depending on where we start, Corollary 4.34 implies that we either approach the closure $\overline{\lambda}_1$ of a leaf $\widetilde{\lambda}_1 \subset \widetilde{\Lambda}_+$ from the left or the closure $\overline{\lambda}_2$ of a leaf $\widetilde{\lambda}_2 \subset \widetilde{\Lambda}_+$ from the right (see Figure 1).

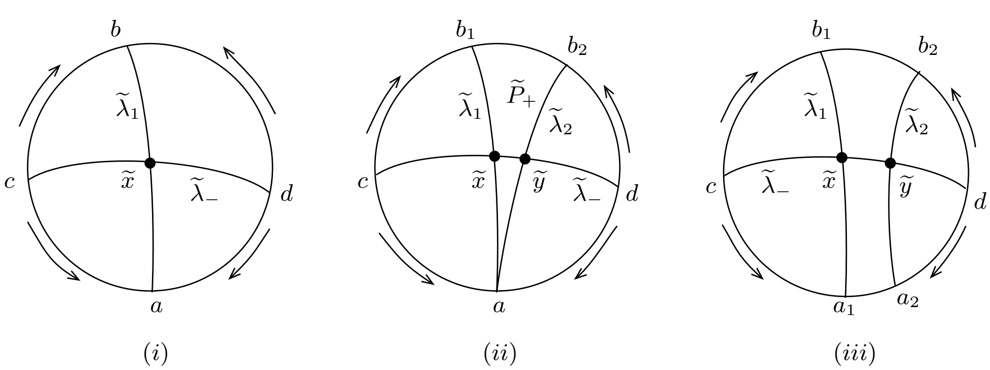

FIGURE 1. Three Cases

Proposition 6.11. *Let $\widetilde{\lambda}_-$ and its ends be fixed by h^p. Then,*

(*i*) *If $\widetilde{\lambda}_1 = \widetilde{\lambda}_2$, the endpoints a, b of $\widetilde{\lambda}_1$ on S^1_∞ are attracting fixed points of $\widetilde{h}^p$ and the endpoints c, d of $\widetilde{\lambda}_-$ on S^1_∞ are repelling fixed points and no point of $S^1_\infty \setminus \{a, b, c, d\}$ is fixed by $\widetilde{h}^p$. The point $\widetilde{x} = \widetilde{\lambda}_1 \cap \widetilde{\lambda}_-$ is fixed by $\widetilde{h}^p$.*

(*ii*) *If $\widetilde{\lambda}_1 \neq \widetilde{\lambda}_2$, the endpoints c, d of $\widetilde{\lambda}_-$ are repelling fixed points of $\widetilde{h}^p$, there is a common endpoint a of $\widetilde{\lambda}_1, \widetilde{\lambda}_2$ which is is an attracting fixed point, and the other endpoints b_1, b_2 of $\widetilde{\lambda}_1, \widetilde{\lambda}_2$ are attracting on the sides facing the endpoints of $\widetilde{\lambda}_-$. No point in the intervals $(c, b_1), (c, a), (d, b_2), (d, a)$ is fixed by $\widetilde{h}^p$. The points $\widetilde{x} = \widetilde{\lambda}_1 \cap \widetilde{\lambda}_-$ and $\widetilde{y} = \widetilde{\lambda}_2 \cap \widetilde{\lambda}_-$ are each fixed by $\widetilde{h}^p$. Furthermore, $\widetilde{\lambda}_1$ and $\widetilde{\lambda}_2$ are border leaves to a lift $\widetilde{P}_+$ of a component of $\mathcal{P}_+$.*

Proof. If $\widetilde{\lambda}_1 = \widetilde{\lambda}_2$, we let $\widetilde{x} = \widetilde{\lambda}_- \cap \widetilde{\lambda}_1$. If $\widetilde{\lambda}_1 \neq \widetilde{\lambda}_2$, we let $\widetilde{x} = \widetilde{\lambda}_- \cap \widetilde{\lambda}_1$ and $\widetilde{y} = \widetilde{\lambda}_- \cap \widetilde{\lambda}_2$. If $\widetilde{\lambda}_1 = \widetilde{\lambda}_2$ we have case (*i*). If $\widetilde{\lambda}_1 \neq \widetilde{\lambda}_2$ but $\widetilde{\lambda}_1, \widetilde{\lambda}_2$ share the endpoint $a \in S^1_\infty$, then we have case (*ii*). The final assertion is due to the fact that, by Lemma 5.7, $[\widetilde{x}, \widetilde{y}]$ projects into $\widehat{\mathcal{P}}_+$.

The behavior in Figure 1(*iii*) cannot happen because $\widetilde{\lambda}_-$ is approached on neither side by components of $\widetilde{\mathcal{X}}_+$ and therefore $\widetilde{\lambda}_-$ is not a leaf of $\widetilde{\Lambda}_-$. In fact, if we start with any component of $\widetilde{\mathcal{X}}_+$ which passes close to and below $\widetilde{\lambda}_-$ and iterate $\widetilde{h}^p$ on this component, the endpoints of the iterates of the component will approach the endpoints a_1, a_2 of $\widetilde{\lambda}_1, \widetilde{\lambda}_2$ and therefore the iterates of the component approach a leaf $\widetilde{\lambda} \subset \widetilde{\Lambda}_-$. But this is a contradiction since iterates under h^p of positive junctures do not approach a leaf of Λ_-, but go off to positive ends of L. A parallel argument works if a lift of a positive juncture is close to and above $\widetilde{\lambda}_-$. □

Parallel considerations hold when $\widetilde{\lambda}_-$ is replaced by a leaf $\widetilde{\lambda}_+$ of $\widetilde{\Lambda}_+$.

Note that, in case (*i*), if we start with any lift of a component σ of $\mathcal{X}_+$ which meets $\widetilde{\lambda}_1$ and iterate using $\widetilde{h}^{-p}$, the endpoints of the iterates of the lift $\widetilde{\sigma}$ will approach the endpoints c, d of $\widetilde{\lambda}_-$ and therefore the iterates of the lift $\widetilde{\sigma}$ approach $\widetilde{\lambda}_-$ either from above or below depending on where we start. In case (*ii*), a lift $\widetilde{\sigma}$ meeting $(\widetilde{x}, a)$ must also meet $(\widetilde{y}, a)$ (it cannot meet $\widetilde{\lambda}_-$) and iterates (by $\widetilde{h}^{-p}$) of $\widetilde{\sigma}$ approach $\widetilde{\lambda}_-$ from below.

Note that, in case (*ii*), no lift of a component of a positive juncture meets both $[\widetilde{x}, b_1)$ and $[\widetilde{y}, b_2)$ for the same reason that case (*iii*) cannot happen.

Corollary 6.12. *In case (i), $\widetilde{x}$ is an attracting $\widetilde{h}$-fixed point on both sides and, in case (ii), $\widetilde{x}$ and $\widetilde{y}$ are attracting on the sides not meeting $(\widetilde{x}, \widetilde{y})$. If $\widetilde{\lambda}_-$ is replaced by $\widetilde{\lambda}_+ \subset \widetilde{\Lambda}_+$, these points are repelling.*

Corollary 6.13. *If $\lambda \in \Lambda_\pm$ and p is the least positive integer such that $h^p(\lambda) = \lambda$ and h^p fixes the ends of λ, then λ contains a periodic point of period p or $p/2$.*

6.3. **Counting the Semi-isolated Leaves.** A semi-isolated leaf $\lambda \in \Lambda_\pm$ will be called *special* if $\lambda \cap K$ has a special component. Evidently, there are only finitely many special leaves.

Lemma 6.14. *If $\lambda \in \Lambda_\pm$ is semi-isolated, then $h^k(\lambda)$ is special for some integer k.*

Proof. Assume on the contrary that λ is semi-isolated, but that $h^k(\lambda)$ is not special, $-\infty < k < \infty$. For definiteness, suppose λ is a leaf of Λ_-, modifying the argument appropriately in the alternative case. Then λ contains a side of a positive rectangle

$R \subset K$. Then $h_*(R) \subset \operatorname{int} K$ and so lies properly in a positive rectangle R_1 (since the sides of $h_*(\mathrm{R})$ are not special). By iterating this argument, we obtain an infinite increasing nest

$$R \subset h_*^{-1}(R_1) \subset h_*^{-2}(R_2) \subset \cdots \subset h_*^{-r}(R_r) \subset \cdots,$$

where each R_r is a positive rectangle. Since R_r is rectangular with two edges segments of negative junctures, $h_*^{-r}(R_r)$ is rectangular with two edges segments of negative junctures that lie arbitrarily deep in neighborhoods of negative ends as $r \to \infty$. The increasing union of this nest of rectangles gives an infinite rectangle whose sides are leaves of Λ_- that repeatedly become a uniformly bounded distance apart (the boundary segments that lie in negative junctures in negative ends can be tightened to geodesic segments of uniformly bounded length). Applying Axiom 1, we see that our rectangles all degenerate to arcs in λ, a contradiction. $\square$

Theorem 6.15. *There are finitely many semi-isolated leaves.*

Proof. By the lemma, the h-orbit of every semi-isolated leaf must contain a special edge. Since there are finitely many such edges, there can be only finitely many special semi-isolated leaves, hence only finitely many h-orbits of semi-isolated leaves and each h-orbit must be periodic. $\square$

Corollary 6.16. *Every semi-isolated leaf has a periodic point.*

Proof. Given a semi-isolated leaf λ, there is an integer p so that $h^p(\lambda) = \lambda$, and so Proposition 6.11 implies that λ has a periodic point. $\square$

Corollary 6.17. *For every negative* (*respectively, positive*) *end e of L, there is a leaf of Λ_-* (*respectively, of Λ_+*) *with an end that passes arbitrarily near e* (*see* Definition 4.13).

Proof. Consider the case that e is a negative end, the other case being entirely analogous. By Axiom 4, every neighborhood of e meets Λ_-, hence meets semi-isolated leaves of Λ_-. Since there are only finitely many of these leaves, the assertion follows. $\square$

Since $\mathcal{U}_\mp \cup \mathcal{P}_\pm$ consists of all complementary regions of $\Lambda_\pm$, the semi-isolated leaves of $\Lambda_\pm$ are exactly the border leaves of these regions. In the case of border leaves of $\mathcal{U}_\pm$, it is possible that both sides of the leaf borders this set, in which case the semi-isolated leaf is actually isolated and the natural map $\widehat{\iota} : \partial\widehat{\mathcal{U}}_\pm \to L$ will not be a one-one immersion, but will identify some boundary components pairwise. In the case of $\delta\mathcal{P}_\pm$, this cannot happen.

Lemma 6.18. *Each leaf ℓ of $\delta\mathcal{P}_\pm$ borders $\mathcal{P}_\pm$ on only one side. Thus, $\widehat{\iota} : \partial\widehat{\mathcal{P}}_\pm \to L$ is one-to-one.*

Proof. Indeed, in the case of $\mathcal{P}_+$, negative h-junctures cluster on ℓ on one side. An analogous argument holds for $\mathcal{P}_-$. $\square$

Lemma 6.19. *The sets of border leaves $\delta\mathcal{U}_\pm$ and $\delta\mathcal{P}_\pm$ are h-invariant.*

Proof. An isolated side of a leaf λ borders $\mathcal{U}_\pm$ if and only if a sequence of leaves of $\mathcal{X}_\mp$ accumulates on λ from that side. By the continuity of h on $\Gamma_\pm$ (Axiom 8), this property is h-invariant. $\square$

6.4. **Escaping Ends.** Let λ be a leaf of $\Lambda_\pm$ and ε an end of λ.

Definition 6.20. A ray $(x,\varepsilon) \subset \lambda$ represents an escaping end ε of λ if there is an end e of L such that, for every neighborhood U of e in L, there is $z \in (x,\varepsilon)$ such that $(z,\varepsilon) \subset U$.

By abuse of language, we often call the ray (x,ε) itself an escaping end.

Let e be a positive end and consider the component $\mathcal{U}_e$ of the positive escaping set. Since there are only finitely many ends and, since there are only finitely many border leaves of $\mathcal{U}_e$, we can assume that there is a value of $p \geq 1$ such that h^p takes each such border leaf to itself, preserving its ends. Somewhat altering previous notation, we will write $\mathcal{U}_e = \bigcup_{n=-\infty}^{\infty} B_e^n$, where B_e^0 is a compact, connected submanifold, cobounded by J^0 and $J^1 = h^p(J^0)$, and $\partial B_e^n = h^{np}(\partial B_e^0)$. If one lets $U_e^n = \bigcup_{k=n}^{\infty} B_e^k$ then $\mathcal{U}_e = \bigcup_{n=-\infty}^{\infty} U_e^n$ and $J^n = h^{np}(J^0) = \partial U_e^n$.

Fix a connected lift $\widetilde{\mathcal{U}}_e$ of $\mathcal{U}_e$. The lift $\widetilde{h}^p$ of h^p that we consider can be chosen to take $\widetilde{\mathcal{U}}_e$, hence $\delta\widetilde{\mathcal{U}}_e$, to itself. In additon, we can assume that $\widetilde{h}^p$ fixes a lift $\widetilde{\lambda}_- \subset \delta\widetilde{U}_e$ of a specific border leaf λ_- of $\mathcal{U}_e$. View $\widetilde{\lambda}_-$ as in Figure 1, (i) or (ii). In case (i), interchange the roles of a and b, if necessary, to assume that $\widetilde{\mathcal{U}}_e$ lies below $\widetilde{\lambda}_-$.

Lemma 6.21. *In case (ii), $\widetilde{\mathcal{U}}_e$ also lies below $\widetilde{\lambda}_-$. In both cases, the ray(s) $(\widetilde{x},a)$ (and $(\widetilde{y},a)$) lie in $\widetilde{\mathcal{U}}_e$.*

Proof. Otherwise, the completions of components of $\widetilde{\mathcal{X}}_+$ would accumulate on $\overline{\lambda}_-$ from above as in Corollary 4.34, leading to the same contradiction as in case (iii) of Figure 1. If a ray $(\widetilde{x},a)$ or $(\widetilde{y},a)$ meets $\widetilde{\Lambda}_-$, then iterates of $\widetilde{h}^{-p}$ produce leaves of $\widetilde{\Lambda}_-$ clustering on $\widetilde{\lambda}_-$ from below. □

Pseudo-geodesics $\widetilde{\ell} \subset \Delta$ have compact completions $\overline{\ell}$ in $\overline{\Delta}$. By Proposition 5.2 and Corollary 4.34, we conclude the following.

Corollary 6.22. *Completions of components of $\widetilde{\mathcal{X}}_+$ accumulate on $\overline{\lambda}_-$ from below, becoming uniformly close in the Euclidean metric on $\overline{\Delta}$.*

By this corollary and Axiom 7, let a component of $\widetilde{\mathcal{X}}_+$ meet $(\widetilde{x},a)$ in the single point u. In case (ii), it also meets $(\widetilde{y},a)$ in a singleton u'. Set $v,v' = \widetilde{h}^p(u), \widetilde{h}^p(u')$.

Lemma 6.23. *Only finitely many components of $\widetilde{\mathcal{X}}_+$ meet the arc $[u,v]$ (and $[u',v']$).*

Proof. Otherwise the intersections of these curves with $[u,v]$ or $[u',v']$ cluster there, implying that $(\widetilde{x},a) \cap \widetilde{\Lambda}_- \neq \emptyset$. This contradicts Lemma 6.21. □

Corollary 6.24. *There exist k_1, k_2 so that if a lift of a component of J^k meets $[u,v]$ (and, if pertinent, $[u',v']$), then $k_1 \leq k \leq k_2$.*

Corollary 6.25. *If a lift of a component of J^k meets $\widetilde{h}^{np}\big([u,v]\big)$ (and, if pertinent, $\widetilde{h}^{np}\big([u',v']\big)$), then $k_1 + n \leq k \leq k_2 + n$.*

Corollary 6.26. *In part (i) of* Proposition 6.11*, the ray $(\widetilde{x},a) \subset \widetilde{\lambda}_1$ is a lift of an escaping end. In part (ii), the entire cusp bounded by $(\widetilde{x},a)$ and $(\widetilde{y},a)$ escapes.*

Proof. As $k \to \infty$, the junctures J^k converge to the positive end e. By Corollary 6.25, for each arbitrarily large value of k, $(\widetilde{x}, a)$ meets exactly finitely many lifts of any component of J^k. Thus, (x, ε) meets each J^k finitely often, for each arbitrarily large value of k, and the assertion follows. The case of cusps is entirely similar. $\square$

Definition 6.27. An end ε of λ returns to the core if every neighborhood of ε in λ meets the core K. One defines analogously cusps that return to the core.

Lemma 6.28. *If the end ε does not return to the core, ε has a neighborhood $[x, \varepsilon)$ with a lift $[\widetilde{x}, a)$ as in* Corollary 6.26*, hence is an escaping end. The analogous assertion holds for cusps.*

Proof. Let $\lambda_- \subset \Lambda_-$ have an end ε that does not return to the core. (The case of such an end of a leaf of Λ_+ is handled by a completely parallel argument.) This end has a neighborhood lying entirely in $\mathcal{U}_-$ and so λ_- must enter $\mathcal{U}_-$ through a point $z \in \ell \in \delta\mathcal{U}_-$ and never exit $\mathcal{U}_-$ again. We will consider the case that ℓ contains a unique h-periodic point z^*, the other case being handled similarly. (One notes that z cannot lie in the interior of a maximal periodic interval $[x, y]$) since such an interval lies in $\widehat{\mathcal{P}}_-$.)

If $z = z^*$, we are done. Otherwise, consider the points $\{z_n = h^{kn}(z)\}_{n=-\infty}^{\infty}$ with $k > 0$ chosen so that h^k fixes z^* and the ends of ℓ. Then, since z^* is h^k-repelling, $\lim_{n\to-\infty} z_n = z^*$ and the points of the bi-infinite sequence $\{z_n\}$ are strictly monotone increasing in the order structure of ℓ. The points $\{z_n\}_{n=-\infty}^{\infty}$ are in the invariant set $\mathcal{I}$, hence lie in the core, and the sequence has compact closure Z in L. There are two cases to consider, each leading to a contradiction:

Case 1. The sequence $\{z_n | n = 0, 1, 2, \ldots\}$ converges to $w \in \ell$ in the intrinsic real-line topology of ℓ. Then, since the sequence $\{z_n\}$ is monotone in ℓ, $w \neq z^*$ is a periodic point which is a contradiction.

We have $(z, \varepsilon) \subset \lambda_- \cap \mathcal{U}_-$, where $z \in \ell$. Then $h^{kn}(z, \varepsilon) = (z_n, \varepsilon_n) \subset \mathcal{U}_-$ is a neighborhood of an end ε_n of a leaf $h^{kn}(\lambda_-)$ that does not return to the core, $-\infty < n < 0$, and $z_n \in \ell$.

Case 2. The sequence $\{z_n | n = 0, 1, 2, \ldots\}$ has no subsequence converging to $w \in \ell$ in the intrinsic real-line topology of ℓ.

Since Z is compact in the topology of L, the sequence $\{z_n | n = 0, 1, 2, \ldots\}$ has a subsequence, $\{w_r\}_{r=1}^{\infty} \subset Z$, converging to a point $w \in Z$. Let W be a neighborhood of w that is a product neighborhood for both laminations. Without loss, assume the entire sequence $\{w_r\}_{r=1}^{\infty} \subset W$. Let $I \subset W$ be a plaque of Λ_+ through w. By assumption, it is not possible that infinitely many of the w_r lie in I. Thus there exist z_m, z_n, z_q, $0 \leq m, n, q$ and distinct plaques $I_m, I_n, I_q \subset \ell$ of W with $z_m \in I_m$, $z_n \in I_n$, $z_q \in I_q$. Without loss, assume the plaque I_q is between the plaques I_m and I_n. Thus, the neighborhood $(z_q, \varepsilon) \subset \mathcal{U}_-$ meets one of the two plaques $I_m, I_n \subset \ell$ which is a contradiction.

The argument for cusps is entirely similar. $\square$

This result, coupled with Corollary 6.26, completely characterizes the escaping ends and escaping cusps.

Theorem 6.29. *The escaping ends of leaves $\lambda_\pm \subset \Lambda_\pm$ are exactly those represented by rays (z,ε) lying in $\mathcal{U}_\pm$ where z is either the unique periodic point on a leaf of $\delta\mathcal{U}_\pm$ or an endpoint of the unique maximal periodic compact interval on such a leaf. The escaping cusps are similarly characterized where the unique periodic point is replaced by the unique maximal compact periodic interval.*

One notes that there are only finitely many escaping ends and escaping cusps.

6.5. **The Structure of Principal Regions and their Crown Sets.** We consider $\mathcal{P}_+$ and its components P_+, but all arguments and results have parallels for $\mathcal{P}_-$ and P_-. These components are the principal regions and, by Theorem 6.15, there are only finitely many of them.

Fix a choice of $P = P_+$.

Lemma 6.30. *The principal region P cannot contain an entire positive h-juncture J.*

Proof. Indeed, J meets a leaf of Λ_+. □

The components of $P \cap K$ will be negative rectangles (Definition 6.4) or will be special components bounded by finitely many simple closed curves

$$\beta_1 \cup \alpha_1 \cup \beta_2 \cup \cdots \cup \beta_r \cup \alpha_r,$$

where the β_i are extreme arcs of $\Lambda_+ \cap K$, alternating with proper subarcs α_i of positive junctures in ∂K. There is a least integer $k > 0$ such that h^k fixes each of the arcs β_i. Then $h^{-k}(\alpha_i)$ is a segment of positive juncture with endpoints in β_i and β_{i+1}, respectively. Infinite iteration gives a sequence of segments of positive junctures converging to a segment σ_i of a leaf λ_i of Λ_- having endpoints x_i and x_i' on β_i and β_{i+1}, respectively. The arc $\sigma_i = [x_i, x_i'] \subset \lambda_i$ is a maximal compact h-periodic subarc. Indeed, the interior of a bigger one would lie in $\mathcal{P}_+$ by Lemma 5.7, contradicting the fact that β_i and β_{i+1} are subsets of Λ_+. Note that $[x_i, x_i'] \subset K$. The two rays in Λ_+ issuing from x_i and x_i' and passing through $\beta_i \cap \alpha_i$ and $\alpha_i \cap \beta_{i+1}$, respectively, together with the arc $[x_i, x_i'] \subset \lambda_i$, bound a cusp in L with lift as in Figure 1(ii). This cusp will be called an *arm* A_i of $\widehat{P}$ (or, by abuse, of P). We will say that the arms A_i and A_{i+1} are adjacent, noting that $A_{r+1} = A_1$, giving us a finite cycle of pairwise adjacent arms. Examples show that every integer $r \geq 1$ can occur. The case $r = 1$ is a bit special in that the cycle reduces to a single arm, but the reader can see that our discussion adapts to that case also. We will say that the leaves $\gamma_1, \dots, \gamma_{r+1} = \gamma_1$ of Λ_+, where $\gamma_i \supset \beta_i$, is a cycle of boundaries of the principal region. Finally, shorten the arcs β_i to have endpoints x_{i-1}' and x_i, defining a simple closed curve $s = \beta_1 \cup \sigma_1 \cup \beta_2 \cup \cdots \cup \beta_r \cup \sigma_r$. Note that the indices are taken mod r.

Remark that, by our earlier discussion, an arm A_i escapes if and only if λ_i is a border leaf of $\mathcal{U}_+$. Otherwise it returns to the core infinitely often. By symmetry under h, either all arms in a cycle escape, or none do.

Lemma 6.31. *The simple closed loop s meets no leaves of $\Lambda_+ \cup \Lambda_-$ transversely.*

Proof. A leaf of Λ_- could only cross s at an interior point of some β_i and a leaf of Λ_+ at an interior point of some σ_i. But β_i is h-periodic, hence lies in the invariant set $\mathcal{I}$. By Lemma 5.7, the interior of β_i must lie in a negative principal region, hence no leaf of Λ_- can cross s. By a similar argument, no leaf of Λ_+ can cross s. □

Definition 6.32. The closure N of the complement in $\widehat{P}$ of the union of all arms of P will be called the nucleus of the principal region.

There are two cases to consider: either s is homotopically trivial or s is essential.

Lemma 6.33. *If s is homotopically trivial, then $\widehat{P}$ is simply connected, its nucleus being a disk.*

Proof. The loop s separates a cycle of arms from the nucleus of P. If s is homotopically trivial, it bounds a disk D that necessarily lies in the nucleus. Since any point in the nucleus can be joined to any other by a path not crossing the boundary, D must be the entire nucleus. □

We now assume that s is essential and identify the crown set in this case.

Remark that $\widehat{P}$ is one-to-one immersed in L and consider a lift $\widetilde{P}$ of $\widehat{P}$ to the universal cover, $\widetilde{s} \subset \widetilde{P}$ the corresponding lift of s. The cycle $\gamma_0, \gamma_1, \dots, \gamma_r = \gamma_0$ of boundaries lifts to a bi-infinite sequence $\dots, \widetilde{\gamma}_0, \widetilde{\gamma}_1, \dots \widetilde{\gamma}_r, \dots$ and each pair $\widetilde{\gamma}_i, \widetilde{\gamma}_{i+1}$ have a common endpoint z_i on the circle at infinity. Similarly, the cycle of arms lifts to a bi-infinite sequence $\dots, \widetilde{A}_0, \widetilde{A}_1, \dots \widetilde{A}_r, \dots$ There will be a deck transformation ψ, determined by s and taking $\widetilde{\gamma}_i$ to $\widetilde{\gamma}_{i+r}$ and $\widetilde{A}_i$ to $\widetilde{A}_{i+r}$, $-\infty < i < \infty$. This deck transformation fixes distinct points a, b in the circle at infinity which are approached by the endpoints of $\widetilde{\gamma}_i$ as $i \to \pm\infty$.

We construct a pseudo-geodesic $\widetilde{\rho}$ in the universal cover joining a to b, not intersecting the union of $\dots, \widetilde{\gamma}_0, \widetilde{\gamma}_1, \dots \widetilde{\gamma}_r, \dots$ nor $\widetilde{s}$, and invariant under ψ. For this, choose a point $p \in \Delta \smallsetminus (\widetilde{\Lambda}_+ \cup \widetilde{\Lambda}_-)$ that is not in the region subtended by the arc $[a, b] \subset S^1_\infty$ and the lift $\widetilde{s}$. The ψ-orbit of p limits on a and b. Choose a simple arc r from p to $\psi(p)$ that is disjoint from $\widetilde{s}$, meets the lifted laminations, if at all, transversely, and note that the ψ-orbit of r unites to give the desired pseudo-geodesic $\widetilde{\rho}$. The covering map projects $\widetilde{\rho}$ onto a closed curve $\rho \subset L$ homotopic to s. Since it is clear that s does not properly intersect itself, we obtain the following.

Lemma 6.34. *The closed curve ρ is simple and is disjoint from $\Lambda_+ \cup \Lambda_-$.*

Proof. If $\widetilde{\rho}$ properly intersected an image of itself under a covering transformation, so would $\widetilde{s}$, contradicting the fact that s is a simple closed curve. Thus, ρ is also simple. If $\widetilde{\rho}$ meets a leaf of the lifted laminations, it does so transversely, hence $\widetilde{s}$ would do so also, contradicting Lemma 6.31. □

All of this is illustrated in Figure 2, where the boldfaced curve is a lift of s.

Since (in the topology of $\widehat{P}$) $B = s \cup A_0 \cup \dots \cup A_{r-1}$ has infinite cyclic fundamental group generated by s, it is clear that any covering transformation that carries a point of a lift $\widetilde{B}$ of this set to another point of $\widetilde{B}$ must be a power of ψ.

Let $\widetilde{C}$ be the closed region of $\widetilde{L}$ bounded by $\widetilde{\rho} \cup \bigcup_{i=-\infty}^{\infty} \widetilde{\gamma}_i$. and let C be the projection of this region into L, ρ the projection of $\widetilde{\rho}$. .

Lemma 6.35. *$C = \widetilde{C}/\psi \subset \widehat{P}$ and this will be called a crown set of P. The loops $\rho \subset \widehat{P}$ can be chosen so that any two crown sets of P either coincide or are disjoint.*

Proof. Distinct points $x, y \in \widetilde{C}$ cannot be identified by any deck transformation ψ' other than a power of ψ. Indeed, $\psi'(\widetilde{B}) \cap \widetilde{B} = \emptyset$ and it readily follows that $\psi'(\widetilde{C}) \cap \widetilde{C} = \emptyset$, hence $C = \widetilde{C}/\psi$. It also follows that s and ρ cobound an annulus in L. By Lemmas 6.31 and 6.34, the interior of this annulus meets no leaf of $\Lambda_\pm$

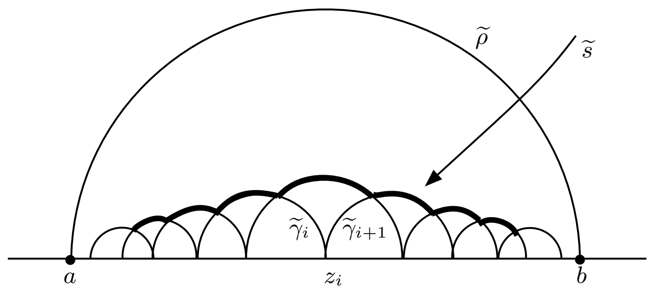

FIGURE 2. The lifts of ρ and s

and it follows that the annulus, hence also C lies in $\widehat{P}$, being cut off from the rest of $\widehat{P}$ by the essential circle ρ.

Now $\widetilde{\rho}$ can be chosen uniformly as close (in the hyperbolic metric) to $\widetilde{s}$ as desired. Just choose p and r close and appeal to the fact that ψ is a hyperbolic isometry and $\widetilde{s}$ is ψ-invariant. It is clear then that suitable choices of the loops ρ in $\widehat{P}$ cut off disjoint crown sets. $\square$

Remark. Each non-simply connected principal region has at least one crown set and so L contains finitely many crown sets. See [2, pp. 63-66] for figures and examples for compact surfaces. Remark that Lemma 6.35 implies the following, which is also clear if P is simply connected.

Lemma 6.36. *The arms of a principal region are simply connected.*

Remark. In Section 11, we will be interested in that portion of an arm of P that is a simply connected component of $K \smallsetminus (\Lambda_+ \cup \Lambda_-)$ which is bounded by an arc of Λ_+, two sides that are arcs of Λ_-, and an arc of ∂K. The sides are in nonisotopic arcs of $\Lambda_- \cap K$. We will call this the "stump" of the arm. It is simply connected.

Definition 6.37. The closed curve $\rho \subset \widehat{P}$ that cuts off a crown set is called a periodic reducing curve. It is also called the *rim* of the crown set.

Remark. In the case that the laminations are geodesic, one can choose the rim ρ to be the unique closed geodesic in its homotopy class.

The rim ρ of a crown set will be included in the set $\mathcal{Z}$ of reducing curves that we are building. Since h permutes the borders of the principle regions, there is a corresponding permutation of the crown sets themselves that we will denote by h_*. Thus, we get cycles $C = C_0, C_1 = h_*(C), \dots, C_n = h_*^n(C) = C_0$, and a corresponding cycle $\rho = \rho_0, \rho_1, \dots \rho_n = \rho_0$ of rims. If ρ is one of these rims and s is the loop described above that is homotopic to ρ, these two loops cobound an annulus A and we can choose h on ρ so that $h|(\rho \cup s)$ extends over A.

Lemma 6.38. *If C is a crown set of a positive principal region P with rim ρ and cycle $\gamma_0, \gamma_1, \dots, \gamma_r = \gamma_0 \subset \Lambda_+$ of bounding leaves, then there is a crown set C' of a negative principal region with the same rim ρ and the same number of bounding*

leaves $\gamma'_0, \gamma'_1, \dots, \gamma'_r = \gamma'_0 \subset \Lambda_-$, *where* γ'_i *intersects* γ_j *iff* $j = i$ *or* $j = i+1$. *Each such intersection is a single point.*

Proof. Consider the bounding leaves γ_i of C and the compact subarcs $\beta_i \subset \gamma_i$, $0 \le i < r$. One endpoint of β_i is joined to one endpoint of β_{i+1} by the arc σ_i in a leaf of Λ_-. To see that this is a border leaf of a negative principal region, proceed as follows. The arc σ_i was obtained as the uniform limit from one side of a sequence of subarcs of positive h-junctures. If the other side of σ_i is not such a limit, then the leaf γ'_i of Λ_- containing σ_i must be a border leaf of a negative principal region. But a sequence of such subarcs of positive h-junctures must have endpoints in β_i and β_{i+1}. Since $h^k(\beta_i) = \beta_i$ and $h^k(\beta_{i+1}) = \beta_{i+1}$, this is impossible. Thus, the arcs β_i cut off arms A'_i of a negative principal region P' and the arcs σ_i lie in corresponding bounding leaves γ'_i for this region. We obtain a crown set C' with the cycle $A'_0, A'_1, \dots, A'_r = A'_0$ of arms and the corresponding cycle $\gamma'_0, \gamma'_1, \dots, \gamma'_r = \gamma'_0 \subset \Lambda_-$ of bounding leaves. Let ρ' be the rim of C'. Lift the set $C \cup C'$ to the universal cover, noting that the points at infinity of the lifted arms $\widetilde{A}_i$ and those of $\widetilde{A}'_i$ alternate in the circle at infinity and so both bi-infinite sequences converge to the same two points. The arc $\widetilde{\rho}$, already constructed, joining these points and missing the laminations descends to the rim of both crown sets, and so $\rho = \rho'$. The remaining assertions are obvious. $\square$

Definition 6.39. The crown sets C and C' are called dual crown sets.

We now consider the nucleus N of $\widehat{P}$. In the case that $\widehat{P}$ is simply connected, N is a disk and there is no crown set. There is a negative, simply connected principal region P' which has the same nucleus $N = \widehat{P} \cap \widehat{P}'$. The reader can check this easily. We will say that P and P' are dual principal regions. The non-simply connected case is analogous, but a bit more subtle.

Lemma 6.40. *Assume that the principal region* P *is not simply connected. The nucleus* N *of* $\widehat{P}$ *is connected, compact, and is identical with the nucleus of the completion* $\widehat{P}'$ *of a unique negative principal region* P', *the crown sets of which are exactly the duals of the crown sets of* P.

Proof. Since $\widehat{P}$ is connected and N is obtained by excising finitely many disjoint arms, it is clear that the nucleus is connected. We claim that N meets no leaves of Λ_- except in the arcs that lie in the loops s corresponding to cycles of arms. Otherwise, Lemma 6.34 implies that an entire leaf of Λ_- lies in N. Such a leaf meets negative junctures, contradicting the fact that $N \subset \widehat{P}$ meets no negative junctures. Of course, N also meets no leaf of Λ_+. If C is a crown set of P, C' the dual crown set pertaining to a negative principal region P', then $C \cap C'$ is an annulus contained in $N \cap N'$, where N' is the nucleus of $\widehat{P}'$. Any arc issuing from the boundary circle s of N in C and staying in N must also stay in N'. This is because any boundary component of N' is made up of arcs of $\Lambda_\pm$ which cannot meet $\operatorname{int} N$. It follows that $N \subseteq N'$ and the reverse inclusion is proven similarly. Thus $N = N'$ and the crown sets of P' are exactly the duals of those of P. Thus $N = N'$ meets no junctures, positive or negative, implying that N is compact. Indeed, since δN is compact, any end of N would be an end of L and its neighborhoods would contain junctures. $\square$

Definition 6.41. The principal regions P and P' are called dual principal regions.

Remark. Case (ii) of Proposition 6.11 gives an alternative way of constructing lifts of simply connected principal regions as well as lifts of the crown sets for non-simply connected ones. We indicate here how this goes without providing complete details. If, in Figure 1, we start with any lift of a component of a positive juncture which meets $(\widetilde{y}, b_2) \subset \widetilde{\lambda}_2$ and iterate using $\widetilde{h}^{-p}$, one endpoint of the iterates of the lift will approach the endpoint d of $\widetilde{\lambda}_-$ and therefore the iterates of the lift approach $\widetilde{\lambda}'_- \subset \Lambda_-$ having one endpoint d and the other in the arc $[b_1, b_2] \subset S^1_\infty$.

One can continue. Apply the same process to $\widetilde{\lambda}'_-$ to obtain $\widetilde{\lambda}_3$ tangent to $\widetilde{\lambda}_2$ at their endpoint b_2 on S^1_∞, etc. Either the process stops when one obtains $\widetilde{\lambda}_n \subset \widetilde{\Lambda}_+$ tangent to $\widetilde{\lambda}_1$ at their common endpoint b_1 on S^1_∞, or it goes on forever. If the process stops one has the case of a simply connected principal region. If the process goes on forever, denote $\widetilde{x}, \widetilde{y}$ by $\widetilde{x}_1, \widetilde{y}_2$ and let $\widetilde{x}_i, \widetilde{y}_{i+1}$ be the corresponding points in $\widetilde{\lambda}_i, \widetilde{\lambda}_{i+1}$. In Figure 3, it turns out that the highlighted curve $\widetilde{s} = \cdots \cup \widetilde{\lambda}_1 \cup \widetilde{\lambda}_- \cup \widetilde{\lambda}_2 \cup \cdots$ projects to the loop s for a non-simply connected principal region, hence has well defined endpoints on S^1_∞, and the lift $\widetilde{\rho}$ of ρ (not pictured) will connect these endpoints without meeting $\widetilde{s}$. The cusps in this figure descend to a cycle of arms for the principal region. The crown set will be the union of this cycle of arms and the annulus cobounded by s and ρ. In general, non-simply connected principal regions may have many distinct crown sets.

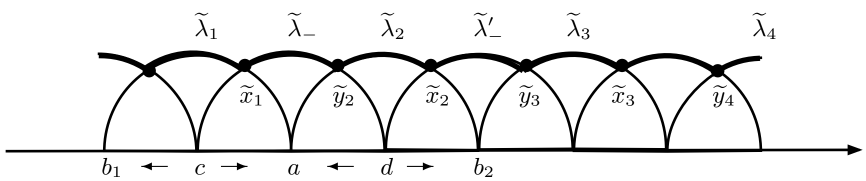

FIGURE 3. Boundary of a Crown Set

7. THE ESCAPING SET $\mathcal{U}$

Recall that the escaping set is defined to be $\mathcal{U} = \mathcal{U}_+ \cap \mathcal{U}_-$. It is not generally connected.

Lemma 7.1. *If U is a component of $\mathcal{U}$, then $\delta U \neq \emptyset$.*

Proof. If $\delta U = \emptyset$ our endperiodic map is a total translation (Lemma 4.2). This case has been excluded. $\square$

There is a certain kind of component U of $\mathcal{U}$ that will play a special role repeatedly:

Definition 7.2. If U is simply connected and δU is a simple closed curve with four distinct edges, all in $\Lambda_+ \cup \Lambda_-$, we will say that U is an escaping rectangle.

We need to understand the components of $K \cap \mathcal{U}$. The components of $K \cap \mathcal{U}_+$ are of two kinds. There are possibly infinitely many "rectangles", bounded by two isotopic arcs of $K \cap \Lambda_-$ and two arcs of negative junctures bordering K. These are positive rectangles (Definition 6.4). There will also be finitely many other components, each bordered by a positive juncture and pieces of negative junctures

and "extreme" arcs of $K \cap \Lambda_-$. We will call these "positive special components". Similarly, one has negative rectangles and negative special components.

The components of $K \cap \mathcal{U} = K \cap \mathcal{U}_+ \cap \mathcal{U}_-$ will be of the following types:

(1) Components of the intersections of positive special components with negative special components. There are finitely many of these, called "special components".
(2) Components of the intersections of positive rectangles and negative rectanges. These are examples of escaping rectangles.
(3) Components of the intersections of positive special components with negative rectangles. These can be of three subtypes:
 (i) The full negative rectangle.
 (ii) An escaping rectangle.
 (iii) A "negative semi-rectangle", this being a simply connected region in $\mathcal{U}$ bordered by an arc of a positive juncture, arcs of two isotopic components of $K \cap \Lambda_+$ and an arc of a component of $K \cap \Lambda_-$.
(4) Components of the intersections of negative special components with positive rectangles. These fall into three types as above, the third being called a "positive semi-rectangle".

Lemma 7.3. *If R is a positive (respectively, negative) semi-rectangle, then $h_*(R) \subset \operatorname{int} K$ (respectively, $h_*^{-1}(R) \subset \operatorname{int} K$).*

Proof. The proof is similar to, but not quite so simple as, that of Lemma 6.6. Let R be a positive semi-rectangle and let σ be the boundary edge that is an arc in a leaf of Λ_+. In order to prove that $h(\partial R) \subset \operatorname{int} K$, it will be enough to prove that $h(\sigma) \subset \operatorname{int} K$. Let $R' \subset K$ be the positive rectangle containing R. Since σ meets Λ_- at its endpoints, so does $h(\sigma)$, hence the endpoints of $h(\sigma)$ lie in $\operatorname{int} K$. If $h(\sigma)$ does not lie entirely in $\operatorname{int} K$, some point of $h(\sigma)$ (not an endpoint) lies in a positive juncture $J \subset \partial K$. Thus, $h^{-1}(J)$ meets $\operatorname{int} R'$. But $h^{-1}(J)$ cannot meet $\partial R'$, hence this entire h-juncture lies in R', a simply connected region, clearly contradicting the fact that all components of h-junctures are essential loops. □

7.1. **The Border of the Escaping Set.**

Lemma 7.4. *The set theoretic boundary $\partial\mathcal{U}$ is a subset of $\Lambda_+ \cup \Lambda_-$, hence $\delta\mathcal{U}$ is contained in the subset of $\Lambda_+ \cup \Lambda_-$ that is the union of the finitely many semi-isolated leaves.*

Proof. A point x is in $\partial\mathcal{U}$ iff $x \notin \mathcal{U}$ but is approached by a sequence $\{x_i\}_{i=1}^\infty \subset \mathcal{U}$. Thus, either $x \notin \mathcal{U}_+$ but all $x_i \in \mathcal{U}_+$ or we have the parallel statement for $\mathcal{U}_-$. In the first case, $x \in \partial\mathcal{U}_+ = \Lambda_-$ and, in the second, $x \in \partial\mathcal{U}_- = \Lambda_+$. Clearly $\delta\mathcal{U} \subset \partial\mathcal{U}$ and, by Corollary 4.38, each of its points must lie in a semi-isolated leaf. □

If $x \in \delta\mathcal{U}$, it lies in some semi-isolated leaf λ of one of the laminations. We consider the two cases: (1) $x \in \Lambda_+ \cap \Lambda_-$ and (2) $x \notin \Lambda_+ \cap \Lambda_-$.

In Case (1), x also lies in a semi-isolated leaf λ' of the other lamination and there are maximal, nondegenerate subarcs $[x, y)$ and $[x, z)$ of λ and λ', respectively, that meet no other points of $\Lambda_+ \cap \Lambda_-$. If y is an end of λ, then $[x, y) \subset \delta\mathcal{U}$ and otherwise $[x, y] \subset \delta\mathcal{U}$ and $y \in \Lambda_+ \cap \Lambda_-$. In either case, the resulting arc is called an *edge* of $\delta\mathcal{U}$ and x is called a *vertex* of $\delta\mathcal{U}$. Similar considerations hold for $[x, z)$.

In Case (2), there is a maximal open subarc $(y, z) \subset \lambda$ containing x and not meeting $\Lambda_+ \cap \Lambda_-$. If one or both of y, z is finite, it lies in $\Lambda_+ \cap \Lambda_-$ and is again

called a vertex of $\delta\mathfrak{U}$, the resulting closed or half-closed subarc of λ being an edge. All of λ cannot be an edge because, as we have seen, a semi-isolated edge borders either a component of $\mathfrak{U}_\pm$ or a principal region on any isolated side. In either case, λ contains an h-periodic point through which passes a leaf of the opposite lamination. Such a leaf cannot meet $\mathfrak{U}$, giving the needed contradiction.

Since $\delta\mathfrak{U}$ may not be a one-one immersion of $\partial\widehat{\mathfrak{U}}$, we need to determine what sort of identifications are possible. While an edge might border $\mathfrak{U}$ on both sides, we have the following.

Lemma 7.5. *Two edges of $\delta\mathfrak{U}$ meeting at the same vertex cannot form a curve bordering $\mathfrak{U}$ on both sides.*

Indeed, the contrary would imply that a leaf of each lamination would meet $\mathfrak{U}$.

Lemma 7.6. *If one edge $\alpha \subset \Lambda_\pm$ of $\delta\mathfrak{U}$ borders $\mathfrak{U}$ on both sides (a repeated edge) and if x is a vertex of α, then from x there emanate either two or three more distinct edges. In either case, two new edges lie in a common leaf of $\Lambda_\mp$ and, in the second case, the third new edge lies in the same leaf of $\Lambda_\pm$ as α.*

Indeed, by Lemma 7.5, there must be at least two new distinct edges. They must both lie in leaves of $\Lambda_\mp$. If they did not belong to a common leaf, either two distinct leaves of $\Lambda_\mp$ would intersect or one leaf would intersect itself transversely. For the same reason, if there is a third new edge, it must lie in the same leaf as α. Note that, if there are three new edges emanating from x, either three or all four of the local quadrants cut off in a neighborhood of x by the intersecting leaves pertain to $\mathfrak{U}$.

Finally, the following is a possibility.

Lemma 7.7. *If $\delta\mathfrak{U}$ has a repeated vertex x but not a repeated edge emanating from x, then exactly four edges emanate from x, two lying in a leaf $\lambda_+ \subset \Lambda_+$ and two in a leaf $\lambda_- \subset \Lambda_-$. Of the four local quadrants cut off in a neighborhood of x by λ_+ and λ_-, exactly two diagonally opposed quadrants pertain to $\mathfrak{U}$.*

Lemma 7.8. *More than four distinct edges of $\delta\mathfrak{U}$ cannot share a common vertex.*

Indeed, the contrary would imply that either two leaves of the same lamination intersected transversely or that one leaf self-intersected transversely.

The following is an easy consequence of Lemma 6.19.

Lemma 7.9. *The set $\delta\mathfrak{U}$ is h-invariant.*

The following is elementary.

Lemma 7.10. *If $\gamma \in \delta\mathfrak{U}$, there is a normal neighborhood of γ on the side bordering $\mathfrak{U}$ that does not meet $\Lambda_\pm \smallsetminus \gamma$ and only meet components of $\mathfrak{X}_\pm$ that cross γ transversely.*

In thinking of a border component γ of $\mathfrak{U}$, it is usually better to think of a boundary component $\widehat{\gamma} \subset \partial\widehat{\mathfrak{U}}$ that is carried onto γ by $\widehat{\imath}$. Repeated edges then appear twice as do repeated vertices. The edges in $\widehat{\gamma}$ can safely be thought of as arcs in leaves of the laminations and the vertices as points of intersection of leaves. From the above discussion, it is clear that exactly two edges of $\widehat{\gamma}$ emanate from one vertex. The following is then trivial if one thinks of $\widehat{\gamma}$.

Lemma 7.11. *Each border component γ in $\delta\mathfrak{U}$ is either an immersed copy of the real line or an immersed circle with an even number of edges.*

Lemma 7.12. *If the component $\gamma \subset \delta\mathcal{U}$ is an immersed line then either its sequence of vertices $\dots, x_i, x_{i+1}, \dots$ is bi-infinite or γ has only one vertex x_0 connecting two unbounded edges α_1 and β_1. In the latter case, γ is one-one immersed and borders $\mathcal{U}$ on only one side. In this case, x_0 is the unique h-periodic point on the semi-isolated leaves of $\Lambda_\pm$ passing through it.*

Proof. Again, think of $\widehat{\gamma}$. Suppose, for definiteness, that the sequence is not infinite to the left and denote its initial vertex by x_0. Thus, its initial edge α_1 must be a ray in a semi-isolated leaf λ of one of the laminations. By Theorem 6.15, we know that, for some integer $k \geq 1$, $h^k(\lambda) = \lambda$. Without loss of generality, we can suppose that h^k fixes the ends of λ. Thus, orienting λ so that its initial end is the end of α_1, we see that either $h^k(x_0) < x_0$, $h^{-k}(x_0) < x_0$, or $h^k(x_0) = x_0$. The first two cases imply that $\operatorname{int}\alpha_1$ meets the other lamination, hence contains a vertex. Hence the third case holds. Now suppose that the edge $\beta_1 = [x_0, x_1]$ is bounded. It must be fixed by h^k and so (x_0, x_1) lies in a principal region (Lemma 5.7), contradicting that this arc lies in $\delta\mathcal{U}$. The fact that γ is one-one immersed should be clear and that it borders $\mathcal{U}$ on only one side is given by Lemma 7.5. The final assertion is also clear. □

Definition 7.13. A real line component of $\delta\mathcal{U}$ with just one vertex is said to be of the first kind. Otherwise, the component is of the second kind.

7.2. **Compact Components of $\delta\mathcal{U}$.** Let C be an element of $\delta\mathcal{U}$ which is an immersed circle. Think of the corresponding component $\widehat{C} \subset \partial\widehat{\mathcal{U}}$. The component U of $\mathcal{U}$ with $\widehat{C} \subset \partial\widehat{U}$ may be bounded or not and it may be simply connected or not.

Definition 7.14. Write $C_n = h^n(C)$, $-\infty < n < \infty$, and call the bi-infinite sequence $\mathcal{C} = \{C_n\}_{n=-\infty}^{\infty}$ a *family* of bounded components of $\delta\mathcal{U}$.

We will prove that there are only finitely many families of such bounded components that do not border escaping rectangles (Definition 7.2). In the following proof, keep in mind the classification on page 39 of the components of $K \cap \mathcal{U}$.

Proposition 7.15. *There are only finitely many families $\mathcal{C}$ whose elements are not the borders of escaping rectangles.*

Proof. For each $n \in \mathbb{Z}$, let U_n be the component of $\mathcal{U}$ having C_n as a border component. For some $C_n \in \mathcal{C}$, suppose that C_n contains the sides of a positive semi-rectangle $R \subset K \cap U_n$ but that, for every $m \geq n$, C_m does not meet any sides of a special component of $K \cap U_m$. Then $h_*(R) \subset \operatorname{int} K$ (Lemma 7.3) and so lies in a component R_1 of $K \cap U_{n+1}$. This cannot be special and, since $h_*(R)$ has two edges that are arcs in leaves of Λ_- and one that is an arc in a leaf of Λ_+, R_1 must be a positive semirectangle. Note that $R \subset h_*^{-1}(R_1)$ and that, by iterating this argument, we obtain an infinite increasing nest

$$R \subset h_*^{-1}(R_1) \subset h_*^{-2}(R_2) \subset \cdots \subset h_*^{-r}(R_r) \subset \cdots,$$

where each R_r is a positive semi-rectangle. Since R_r is rectangular with fourth edge a segment of negative juncture, $h_*^{-r}(R_r)$ is rectangular with fourth side a segment of negative juncture that is arbitrarily deep in neighborhoods of negative ends as $r \to \infty$. The other sides of this rectangle are in C_n and two of them are unbounded, clearly a contradiction. Thus, some C_m, $m \geq n$, meets sides of a special component of $K \cap U_m$.

Similarly, if some $C_n \in \mathcal{C}$ contains the sides of a negative semi-rectangle $R \subset K \cap U_n$, there is $m \leq n$ such that C_m meets sides of a special component of $K \cap U_m$.

If no $C_n \in \mathcal{C}$ contains the sides of a positive or negative semi-rectangle, we can again prove that some C_m meets sides of a special component of $K \cap U_m$. If C_n contains a side of a positive or negative rectangle, but no C_m meets sides of a special component of $K \cap U_m$, a similar argument to the above proves that an edge of C_n is a full leaf of one of the laminations.

Since there are only finitely many special components of $K \cap \mathcal{U}$, each with finitely many border edges, and since some element of each family $\mathcal{C}$ contains one of those edges, it follows that there can be only finitely many of these families. $\square$

Let $C \in \delta\mathcal{U}$ be a circle component bounding a component U that is not an escaping rectangle. Write $C = \alpha_1 \cup \beta_1 \cup \alpha_2 \cup \cdots \cup \beta_r$, where the α_i's are segments of leaves of Λ_- and the β_i's are segments of leaves of Λ_+. Set $\alpha_{r+1} = \alpha_1$ and $\beta_{r+1} = \beta_1$. In $\widehat{U}$ there are segments of negative h-junctures joining α_i to α_{i+1} and accumulating on β_i. Since U does not meet Λ_+ and is not an escaping rectangle, there is an extreme (i.e., furthest from β_i) such segment δ_i. Similarly, there is an extreme segment $\varepsilon_i \subset \widehat{U}$ of positive juncture. The segments intersect sequentially and the resulting subsegments unite to form a closed loop σ in U that cobounds an annulus with C. Evidently, σ meets $\mathcal{X}_\pm$ transversely except that each of its edges lies in a juncture.

We assume that C is essential, in which case σ will be taken to be a component of the reducing set $\mathcal{Z}$. Clearly h is already defined on σ and $h(\sigma)$ is also an essential simple closed curve missing $\Lambda_\pm$ and meeting $\mathcal{X}_\pm$ transversely. Continuing in this way, we obtain a family $\{h^n(\sigma)\}_{n=-\infty}^{\infty} \subseteq \mathcal{Z}$.

If h were defined on all of L, the proof of the following lemma would be practically immediate. This is one of the places where our choice not to first reduce and then extend h in each reduced piece causes some awkwardness.

Lemma 7.16. *The circle σ escapes to positive and negative ends under forward and backward iterations of h.*

Proof. At most a finite number of junctures meet σ since, otherwise, these junctures would accumulate on leaves of $\Lambda_\pm$ meeting U. Evidently, the same number meet each $h^n(\sigma)$. A segment of σ from $\mathcal{X}_-$ evidently escapes to negative ends. Its endpoints also escape to positive ends. If the segment itself did not escape to positive ends, it would stretch under forward iterations of h so as to intersect arbitrarily many positive junctures. Thus the segment escapes and an entirely parallel argument shows that the segments of σ from $\mathcal{X}_+$ also escape. $\square$

Remark. In the case that the laminations are geodesic and C is essential, one easily sees that σ can be replaced by the unique closed geodesic homotopic to C.

Remark. As one forwardly iterates applications of h to C, the vertices remain in K and the edges α_i stretch without bound. Similarly, under iterates of h^{-1}, the edges β_i become unbounded. The fact that the circles σ escape is, therefore, somewhat surprising. We leave the proof of these assertions as an exercise.

7.3. Components of $\delta\mathcal{U}$ of the First Kind.

Proposition 7.17. *There are only finitely many immersed line components $\gamma \subset \delta\mathcal{U}$ of the first kind and, for each, there exists an integer k such that $h^k(\gamma) = \gamma$.*

Indeed, the periodicity follows from our proof of Lemma 7.12, where it was shown that the vertex $x_0 \in \gamma$ is the unique h-periodic point on the semi-isolated leaves through that point. Since there are only finitely many semi-isolated leaves in $\Lambda_\pm$, the finiteness of the number of γ of the first kind is also obvious.

Write $\gamma = \alpha_1 \cup \beta_1$ where α_1 is a ray in a leaf of Λ_- and β_1 is a ray in a leaf of Λ_+, meeting at the unique vertex y_1, and let $\widetilde{\alpha}_1$, $\widetilde{\beta}_1$ be lifts of α_1, β_1 to the universal cover such that $\widetilde{\alpha}_1 \cap \widetilde{\beta}_1 = \widetilde{y}_1$, a lift of y_1. Let $a, b \in S^1_\infty$ be such that a is the negative end of $\widetilde{\alpha}_1$ and b is the positive end of $\widetilde{\beta}_1$. There is a normal neighborhood N on one side of $\widetilde{\gamma}$, the interior of which does not meet $\widetilde{\Lambda}_\pm$ and N only meets $\widetilde{\mathcal{X}}_\pm$ in leaves that cross $\widetilde{\gamma}$ transversely. Let $\widetilde{\sigma}$ be a curve in the Poincaré disk with endpoints a and b, lying in N and transverse to $\widetilde{\mathcal{X}}$.

By Theorem 6.29, the ends of α_1 and β_1 escape. One can choose $\widetilde{\sigma}$ so close to $\widetilde{\gamma}$ that its projection $\sigma \subset L$ has both ends escaping. This will be one of a finite family of reducing curves in $\mathcal{Z}$ corresponding to the finitely many components of $\delta\mathcal{U}$ of the first kind. In order to define h on these curves, permuting them exactly as it permutes the corresponding components of $\delta\mathcal{U}$, we must modify h harmlessly on some of the h-junctures. Indeed, if $\sigma_1, \dots, \sigma_k = \sigma_1$ correspond to an h-cycle of components of $\delta\mathcal{U}$ of the first kind, one easily rechooses these curves so that h extends to a map that carries σ_i to σ_{i+1}, $1 \le i \le k-2$. For σ_{k-1}, it will be necessary to modify the definition of h on segments of junctures that lie in $\mathcal{U}$ so that h carries the intersection points of σ_{k-1} with these junctures to the intersection points of σ_1 with the image junctures. But these segments do not accumulate anywhere in L, so no continuity problems arise. Thus h can be assumed to be defined on that portion $\mathcal{Z}'$ of the reducing curves $\mathcal{Z}$ so far defined so as to be a homeomorphism of $\Gamma_+ \cup \Gamma_- \cup \mathcal{Z}'$ onto itself.

Lemma 7.18. *The component O of $L \setminus \gamma \setminus \sigma$ with $\delta O = \gamma \cup \sigma$ is homeomorphic to $(0,1) \times \mathbb{R}$.*

Indeed, this is true for the lift $\widetilde{O} = \operatorname{int} N$ bounded by $\widetilde{\gamma} \cup \widetilde{\sigma}$ and, if any point of $\widetilde{O}$ were carried to another by a nontrivial covering transformation, it would follow that $\operatorname{int} N$ meets $\widetilde{\Lambda}_\pm$.

Lemma 7.19. *If σ is as above and h^p carries σ to itself while fixing the two ends of σ, then $h^p|\sigma$ has no fixed point and translates σ from the negative end to the positive end.*

Proof. Indeed, the segments of negative junctures in $\widehat{O}$ have endpoints in α_1 and σ. While the endpoints on α_1 cluster on the vertex y_1 under positive iterates of h^p, the endpoints on σ cannot cluster in σ under these iterates. Otherwise, the junctures would cluster on a leaf of Λ_+ meeting O. Of course, under negative iterates of h^p, the endpoints diverge to $-\infty$. This is enough to prove the assrtion, but corresponding assertions hold for segments of positive junctures and Figure 4 depicts, for future reference, both sets of juncture segments as dotted lines. □

Remark. In the case that the laminations are geodesic, one easily sees that $\widetilde{\sigma}$ can be chosen to be the unique geodesic joining $a = \widetilde{\alpha}_1 \cap S^1_\infty$ to $b = \widetilde{\beta}_1 \cap S^1_\infty$

7.4. **Components of $\delta\mathcal{U}$ of the Second Kind.** Let $\gamma = \bigcup_{i=-\infty}^{\infty} \alpha_i \cup \beta_i \subset \delta\mathcal{U}$ be a component of the second kind, $\alpha_i \subset \Lambda_-$, $\beta_i \subset \Lambda_+$. Set $\alpha_n \cap \beta_{n-1} = x_n$ and $\alpha_n \cap \beta_n = y_n$.

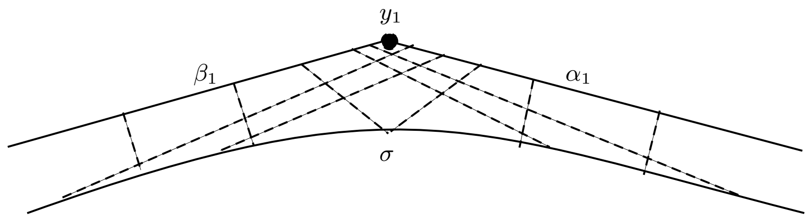

Figure 4. The segments of junctures in $\widehat{O}$

Definition 7.20. An edge of γ is a *special edge* if it is a border arc of a special component of $K \cap \mathcal{U}$.

Theorem 7.21. *Every component γ of the second kind is h-cyclic.*

Proof. We let $\gamma_n = h^n(\gamma)$ and consider three cases.

Suppose that γ has edges $\alpha_i \subset \Lambda_-$ that penetrate arbitrarily deeply towards a negative end of L. Choose $n > 0$ so that γ_n meets W^+. Such an integer n must exist since, otherwise, every β_i would be in the invariant set, hence in a principal region (Lemma 5.7). Then, γ_k meets both W^- and W^+ for all $k \geq n$ and it follows that, for all $k \geq n$, γ_k has special edges. Since there are finitely many special edges, the set $\{\gamma_k | k \geq n\}$ is a finite set and there exists p so that $\gamma_{n+p} = \gamma_n$.

A similar argument handles the case that γ has edges $\beta_i \subset \Lambda_+$ that penetrate arbitrarily deeply towards a positive end.

The remaining case is the one in which γ has no sequence $\alpha_i \subset \Lambda_-$ of edges that penetrates arbitrarily deeply towards a negative end and no sequence $\beta_i \subset \Lambda_+$ that penetrates arbitrarily deeply towards a positive end. By taking the core larger if necessary we can assume that γ remains in the core. But by our classification of components of $K \cap \mathcal{U}$ (page 39), no component of $\delta\mathcal{U}$ that lies entirely in the core can have infinitely many edges. $\square$

Corollary 7.22. *There are only finitely many components of the second kind.*

Proof. These components fall into a disjoint union of h-cycles, each of which contains an element with a special edge. There are only finitely many special edges. $\square$

Let $\widetilde{\alpha}_n$, $\widetilde{\beta}_n$ be lifts of α_n, β_n to the universal cover such that $\widetilde{\alpha}_n \cap \widetilde{\beta}_{n-1} = \widetilde{x}_n$, $\widetilde{\alpha}_n \cap \widetilde{\beta}_n = \widetilde{y}_n$ with $\widetilde{x}_n$, $\widetilde{y}_n$ lifts of x_n, y_n, $-\infty < n < \infty$. These arcs unite to give a lift $\widetilde{\gamma}$ of γ. If $h^p\gamma = \gamma$, we replace h with h^{2p}, if necessary, so as to assume that $h(\gamma) = \gamma$ and h fixes the ends of γ. This simplifies notation. For a suitable lift $\widetilde{h}$, we have $\widetilde{h}(\widetilde{\gamma}) = \widetilde{\gamma}$ and $\widehat{h}$ fixes the ends of $\widetilde{\gamma}$.

In Figure 5, we view $\widetilde{\gamma}$ in the upper half plane model. The curve $\widetilde{\sigma}$ in that picture will be discussed shortly. The fact that $\widetilde{\gamma}$ looks like this is easily seen because of this curve being a border component of a lift $\widetilde{\mathcal{U}}$ of $\mathcal{U}$. Actually, Figure 6 is also possible, but viewing these figures in the unit disk model makes clear that they are equivalent. In the following discussion, we think in terms of Figure 5.

Let a_i, a_i^* be the endpoints on the circle at infinity of the leaf of the lamination containing $\widetilde{\alpha}_i$. Let b_i, b_i^* be the endpoints on the circle at infinity of the leaf of the lamination containing $\widetilde{\beta}_i$. Let $A = \{a_i, a_i^*, b_i, b_i^* \mid i \in \mathbb{Z}\}$.

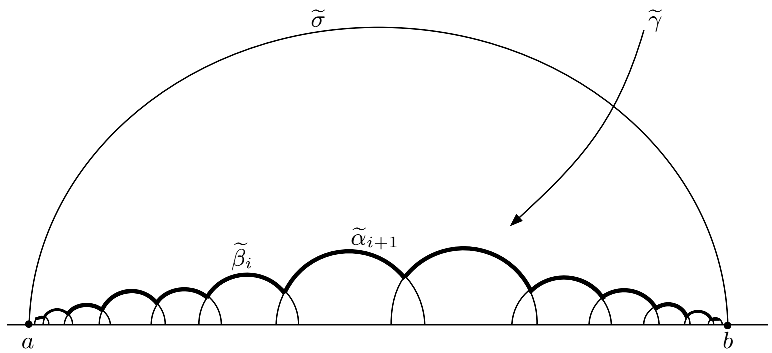

FIGURE 5. The lift $\widetilde{\gamma}$ and the arc $\widetilde{\sigma}$ – first possibility

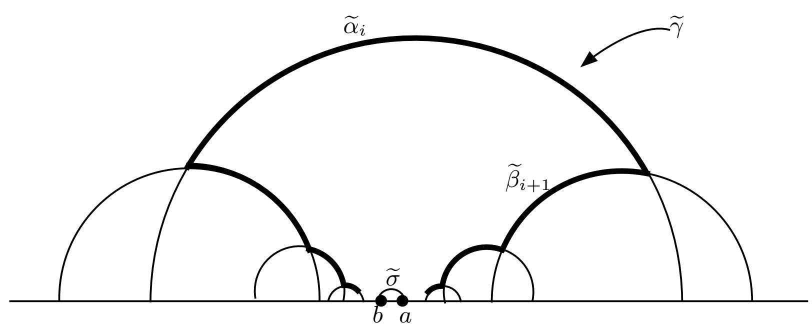

FIGURE 6. The lift $\widetilde{\gamma}$ and the arc $\widetilde{\sigma}$ – second possibility

Lemma 7.23. *$\widehat{h}(A) = A$.*

The set A comes with a total order in Figure 5. Let $a = \inf A$ and $b = \sup A$. It is possible that $a = -\infty = \infty$ and/or $b = \infty$. If only one of these happens, a suitable linear fractional transformation puts us in the case that a and b are both finite and distinct (as assumed in Figure 5). We will see that this is the case.

Consider all lifts of components of negative h-junctures that cross $\widetilde{\alpha}_i$'s and remark that they are of two types. Those that again cross some $\widetilde{\alpha}_j$ will be called lifts of the first type. Those that do not will be of the second type. We also consider lifts of components of positive junctures that cross $\widetilde{\beta}_i$'s and define the two types analogously.

Lemma 7.24. *Not all lifts, negative or positive, that cross $\widetilde{\gamma}$ are of the first type.*

Proof. Suppose that all lifts of components of negative junctures that cross $\widetilde{\gamma}$ are of the first type. Fix $j > i$ and consider those lifts with an arc I_k issuing from $u_k \in \alpha_i$, terminating at $v_k \in \alpha_j$, and meeting no other point of $\widetilde{\gamma}$. Ordering the points of

$\widetilde{\gamma}$ from a to b, we can define $I_k < I_l$ to mean that $u_l < u_k$ and $v_l > v_k$. There is an extreme $I_n > I_k$, $\forall k$, since otherwise the I_k's would accumulate on an arc of a leaf of $\widetilde{\Lambda}_+$ crossing α_i and α_j. As i and j vary, these extreme arcs are necessarily disjoint and, together with subarcs of the α_i's, form an imbedded line ℓ in $\widetilde{L} = \Delta$ with endpoints a and b on S^1_∞. The interior of a one-sided neighborhood O of ℓ meets no lifted component of a negative juncture. Let I be an extreme arc and construct a path τ in O that issues from I, terminates on $\widetilde{h}^2(I)$, and otherwise lies in $\operatorname{int} O$. The projection of τ into L issues from a negative juncture J, terminates on $h^2(J)$, but cannot cross $h(J)$, a clear contradiction. Similarly, not every lifted positive juncture that crosses $\widetilde{\gamma}$ can be of first type. □

Consider the two components W and V of the complement of $\widetilde{\gamma}$ in the open unit disk, where V is the component bordered by $\widetilde{\gamma}$ on the same side that this curve borders a lift of $\mathcal{U}$.

Corollary 7.25. $a \neq b$.

Proof. Since components of junctures are simple closed curves in L, one that crosses γ into $\mathcal{U}$ must also exit $\mathcal{U}$ through $\delta\mathcal{U}$. In the universal cover, the assumption that $a = b$ implies that no leaf of $\widetilde{\Lambda}_\pm$ meets V (otherwise, such a leaf would cross $\widetilde{\gamma}$), hence $V \subset \widetilde{\mathcal{U}}$. This clearly implies that the lifted junctures crossing $\widetilde{\gamma}$ are all of the first type. □

We can now assume that in Figure 5 the points a and b are distinct and finite. We can also assume that $\widetilde{h}$ moves the points of $\widetilde{\gamma}$ away from a and towards b.

Lemma 7.26. *If $\widetilde{\tau}$ is a component of the first type, then $\widetilde{\tau} \cap V \subset \widetilde{\mathcal{U}}$.*

Proof. Indeed, $\widetilde{\tau} \cap V$ meets $\widetilde{\mathcal{U}}$. If it also met a leaf of $\widetilde{\Lambda}_\pm$, then that leaf would have to properly cross $\widetilde{\gamma}$. □

Choose $\widetilde{\sigma}$ in Figure 5 to be a curve joining a to b and missing $\widetilde{\gamma}$ and all lifted junctures of the first type.

Lemma 7.27. *The curve $\widetilde{\sigma}$ can be chosen so that it projects to a curve $\sigma \subset \mathcal{U}$.*

Proof. Indeed, $\widetilde{\sigma}$ can be chosen close enough to the union of $\widetilde{\gamma}$ and lifted components of the first type so as to lie in $\widetilde{\mathcal{U}}$. □

We can assume that the intersection of $\widetilde{\sigma}$ with each lifted component of a juncture is a singleton at which the curves intersect transversely. Thus, $\widetilde{\sigma}$ projects to a curve in L that can be taken as a reducing curve in $\mathcal{Z}$ and we can assume that $\widetilde{h}$ is defined on $\widetilde{\sigma}$, carrying that curve onto itself and fixing a and b.

Lemma 7.28. *The projection $\sigma \subset L$ of the curve $\widetilde{\sigma}$ joins a negative end of L to a positive end.*

Proof. The negative lifted junctures of the second type all cross $\widetilde{\sigma}$. Using these junctures and an argument entirely parallel to that in the proof of Corollary 6.26, we prove that as $\widetilde{\sigma}$ approaches b, its projection downstairs escapes to a negative end of L. Similarly, as $\widetilde{\sigma}$ approaches a, its projection escapes to a positive end. □

Lemma 7.29. *The component O of $L \smallsetminus \gamma \smallsetminus \sigma$ with $\delta O = \gamma \cup \sigma$ is homeomorphic to $(0,1) \times \mathbb{R}$ and $h|\sigma$ is fixed point free, being a translation of σ from the negative end to the positive.*

Since, by the choice of $\widetilde{\sigma}$, $\widetilde{O}$ cannot meet $\widetilde{\Lambda}_\pm$, the proof of the first assertion is the same as for Lemma 7.18. The second is proven similarly to Lemma 7.19, using the lifted components of junctures of the first type.

We now return to the original h that permutes the finitely many components of $\delta\mathcal{U}$ of the second kind. Exactly as in the case of components of the first kind, we can assume that h is also defined on the associated reducing curves σ_i, permuting them appropriately, and is fixed point free on them, moving points from negative ends toward positive ends. There remain a few (closed) reducing curves to be defined in compact submanifolds of the invariant set. At this point, h has been extended to all but these finitely many closed reducing curves. We will let $\mathcal{Z}$ denote the union of reducing curves already defined, subsequently augmenting $\mathcal{Z}$ by the finitely many closed reducing curves yet to be defined.

Remark. In the case that the laminations are geodesic, one can take $\widetilde{\sigma}$ to be the unique geodesic joining b to a. The easy arguments are left to the reader.

Remark. Under iterates of h, the edges α_i stretch unboundedly and under iterates of h^{-1} the edges β_i stretch unboundedly. Thus, these border components of the second kind appear quite bizarre in L and, in no sense, do they connect a negative end of L to a positive one. The fact that they are "shadowed" by a line σ that does connect two such ends is, therefore, surprising. Details are left as an exercise.

8. Extending the Homeomorphism h

In some applications that we have in mind, the following property will be obviously true.

Axiom 9. The homeomorphism $h : \Gamma_+ \cup \Gamma_- \to \Gamma_+ \cup \Gamma_-$ extends to a homeomorphism $h_* : L \to L$.

In the abstract, however, this is not obvious, even in the case that the laminations are geodesic. The goal of this section is to prove the following.

Theorem 8.1. *In the case of geodesic laminations,* Axiom 9 *holds.*

In the case of geodesic laminations, we can assume that, in fundamental neighborhoods of the ends, the f-junctures and h-junctures coincide and are geodesic, f being an isometry in these neighborhoods. These junctures in fundamental neighborhoods of arm have been called "honest" junctures earlier.

Corollary 8.2. *The extension h_* is an endperiodic homeomorphism isotopic to f through endperiodic homeomorphisms.*

Proof. By Axiom 8, the lift $\widetilde{h}_*$ induces $\widehat{h}_* : S^1_\infty \to S^1_\infty$ which coincides with $\widehat{h} = \widehat{f}$. Thus, h_* is isotopic to f. Since f and h_* induce the same permutations of honest junctures, h_* is endperiodic. Now, $g_0 = f^{-1} \circ h_*$ carries each fundamental domain in the ends to itself and, restricting attention to the universal cover of each fundamental domain, we see that $\widehat{g}_0 = \mathrm{id}$ there, hence the restriction of g_0 to each of the fundamental domains is isotopic to the identity. Similarly, $g_0|K$ is isotopic to the identity. Choosing these isotopies to agree on common boundary components, we obtain an isotopy g_t on L that leaves all fundamental domains in the ends invariant, $0 \le t \le 1$, with $g_1 = \mathrm{id}$. Thus, $f \circ g_t$ defines an isotopy of h_* to f through endperiodic homeomorphisms. □

The system $\mathcal{Z}$ of reducing curves that we have defined plays a critically useful role in defining the extension. In the geodesic case, we have remarked that $\mathcal{Z}$ can be taken to be a family of geodesics. The extension will be defined on the components of $L \setminus \mathcal{Z}$, one at a time. If L has resulted from a doubling of an endperiodic surface F (cf. the remark on page 24), the curves making up ∂F are elements of $\mathcal{Z}$. In any event, the elements of $\mathcal{Z}$ are the periodic curves, the curves "shadowing" the essential closed curves in $\delta\mathcal{U}$, and the lines "shadowing" the noncompact components of $\delta\mathcal{U}$ of first and second kind.

8.1. **The Bounded Components.** Some components of $L \setminus \mathcal{Z}$ may be bounded. Let N be the closure of such a component, a compact connected subsurface of L with ∂N a finite union of circle components of $\mathcal{Z}$. Since the closed loops in $\mathcal{Z}$ are all essential, N is not a disk.

Lemma 8.3. *The subsurface $N \subset L$ is disjoint from $\Lambda_\pm$.*

Indeed, no leaf of $\Lambda_\pm$ stays in a bounded region of L (Axiom 4) and $\partial N \subset \mathcal{Z}$ is disjoint from $\Lambda_\pm$.

Remark that $f : N \to L$ is an imbedding and that $f|\partial N \sim h|\partial N$, where "$\sim$" denotes homotopy. (Indeed, since f has a lift $\widetilde{f}$ inducing $\widehat{f} = \widehat{h}$ on S^1_∞, the lifts $\widetilde{\sigma}$ of each component σ of ∂N are carried by $\widetilde{f}$ and $\widetilde{h}$ to curves on Δ, covering $f(\sigma)$ and $h(\sigma)$ respectively, that are homotopic with endpoints fixed.) The following is easily proven by slightly modifying the usual proof of the homotopy extension property.

Lemma 8.4. *The imbedding $f : N \to L$ is homotopic to an imbedding $h_* : N \to L$ such that $h_*|\partial N = h|\partial N$.*

In this way we extend h continuously to a homeomorphism on each bounded component of $L \setminus \mathcal{Z}$ with image such a bounded component. Let $\mathcal{N}$ denote the union of the closures of these bounded components and let

$$h_* : \mathcal{N} \cup \Gamma_+ \cup \Gamma_- \cup \mathcal{Z} \to \mathcal{N} \cup \Gamma_+ \cup \Gamma_- \cup \mathcal{Z}$$

be the extension of h just constructed.

We need that this extension of h has a lift $\widetilde{h}_*$ that agrees with $\widetilde{h}$ on $\widetilde{\Gamma}_+ \cup \widetilde{\Gamma}_- \cup \widetilde{\mathcal{Z}}$. For this we will use the homotopy lifting property for covering spaces.

Note that any two lifts of closed components N and N' as above have disjoint interiors. Otherwise, the interiors of N and N' would meet $\mathcal{Z}$. Thus, the the lift $\widetilde{\mathcal{N}}$ of $\mathcal{N}$ is the universal cover of $\mathcal{N}$.

Let $F : (\mathcal{N} \cup \Gamma_+ \cup \Gamma_- \cup \mathcal{Z}) \times I \to L$ be the homotopy of the restriction of f to this domain with h_*. Define

$$H = F \circ (\pi \times \mathrm{id}) : (\widetilde{\mathcal{N}} \cup \widetilde{\Gamma}_+ \cup \widetilde{\Gamma}_- \cup \widetilde{\mathcal{Z}}) \times I \to L,$$

where $\pi : \widetilde{L} \to L$ is the covering projection. By the homotopy lifting property, the lift $\widetilde{f}$ extends to a lift $\widetilde{H}$ of H, giving a commutative diagram

$$\begin{array}{ccc} (\widetilde{\mathcal{N}} \cup \widetilde{\Gamma}_+ \cup \widetilde{\Gamma}_- \cup \widetilde{\mathcal{Z}}) \times I & \xrightarrow{\widetilde{H}} & \widetilde{L} \\ {\scriptstyle \pi \times \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \pi} \\ (\mathcal{N} \cup \Gamma_+ \cup \Gamma_- \cup \mathcal{Z}) \times I & \xrightarrow[F]{} & L. \end{array}$$

Evidently $\widetilde{H}$ can also be viewed as a lift of F to a homotopy of $\widetilde{f}$ to a lift $\widetilde{h}_*$ which, because the lifts of curves in $\mathcal{Z}$ form an $\widetilde{h}$-invariant family of curves with endpoints

on S^1_∞, induces the homeomorphism $\widehat{f} = \widehat{h}$ on S^1_∞. Thus, $\widetilde{h}_*$ agrees with $\widetilde{h}$ on $\widetilde{\Gamma}_+ \cup \widetilde{\Gamma}_- \cup \widetilde{\mathcal{Z}}$.

Finally, for the projected application of this theory in [5], we consider each compact component N such that $h^p_*(N) = N$, for some least integer $p \geq 1$. This is exactly the case in which ∂N consists of the rims of dual crown sets. Set $N_0 = N = N_p$ and $N_i = h^i_*(N_0)$, $0 \leq i \leq p$. Each $N_i = h^p_*(N_i)$ and the Nielsen-Thurston theory [13, 23, 24] applies to $h^p_*|N_i$, $0 \leq i \leq p$. Thus, we increase $\mathcal{Z}$ by a family of reducing curves in N_0 and their h^i_*-images in N_i, redefining $h_* : N_{p-1} \to N_0$ so that this new family of reducing curves is h_*-invariant. These curves can be assumed to have thin, open normal neighborhoods which partition each N_i into compact subsurfaces S_{ij}, each invariant under $h^{r_{ij}p}_*$, for suitable least positive integers r_{ij}, $0 \leq j \leq r_{ij}$, on which each $h^{r_{ij}p}_*$ is isotopic to a homeomorphism g_{ij} which is either periodic or pseudo-Anosov. We want to change h_* within its isotopy class on $N_0 \cup \cdots \cup N_{p-1}$ so that each $h^{r_{ij}p}_* = g_{ij}$. To simplify notation, set $r_{ij}p = q$, denote the corresponding cycle of S_{ij}'s by $S_0, \ldots, S_{q-1}$ with $g_k : S_k \to S_k$ pseudo-Anosov (respectively, periodic), $0 \leq k < q$.

Lemma 8.5. *The homeomorphism $h_*|(S_0 \cup \cdots \cup S_{q-1})$ is isotopic to a homeomorhism $h_\sharp$ such that $h^q_\sharp|S_k = g_k$, $0 \leq k < q$.*

Proof. We take $h_\sharp|S_k = h_*|S_k$, $0 \leq k < q-1$ and let $h_\sharp = g_0 \circ h_*^{-q+1}$ on S_{q-1}. Clearly, $h^q_\sharp = g_0$ on S_0 and $h^q_\sharp|S_k = h_\sharp^{-k} \circ g_0 \circ h^k_\sharp$, $0 < k < q$, can be taken as g_k, being pseudo-Anosov (respectively, periodic) if g_0 is. □

In our projected applications, when g_k is pseudo-Anosov, we augment our laminations $\Lambda_\pm$ by adding in the Nielsen-Thurston laminations in S_k.

8.2. **Unbounded Components Lying in $\mathcal{U}$.** If Q is a component of $L \smallsetminus \mathcal{Z}$ that lies in $\mathcal{U}$, it is bordered by some of the loops σ associated to closed, essential components of $\delta\mathcal{U}$ and/or the imbedded lines σ associated to components of $\delta\mathcal{U}$ of first and/or second kind. (If $L = 2F$, some of these will be components of ∂F.) The lines each connect a negative end of L to a positive end.

Proposition 8.6. *There are finitely many unbounded components Q as above that lie in $\mathcal{U}$ and h extends to an automorphism h_* on the internal completion of their union. For each such component Q, there is a least integer $p \geq 1$ such that the induced automorphism $h^p_* : \widehat{Q} \to \widehat{Q}$ is an endperiodic map which is a total translation.*

Proof. There are only finitely many components of $\delta\mathcal{U}$ that are immersed lines (Proposition 7.17 and Corollary 7.22) and finitely many families of circle components (Proposition 7.15). One easily concludes that there are only finitely many components Q as considered in this proposition. Thus, the borders of these components are permuted by h in finitely many cycles.

By construction, each point of each component of δQ escapes and meets junctures. If J is an h-juncture meeting Q, it must intersect Q in finitely many properly imbedded arcs with endpoints on δQ (since f is not a total translation). Thus, $J \cap \widehat{Q}$ consists of arcs with escaping endpoints. If any such arc does not escape, suitable iterates of h on that arc accumulate on an arc of $\Lambda_\pm \cap \widehat{Q}$. This is impossible, hence every component of $J \cap \widehat{Q}$ escapes, for every h-juncture J meeting Q. Evidently, the closures of the components of the complements of $Q \smallsetminus \mathcal{X}$ are compact and each

has escaping boundary. As in the previous subsection, h extends to h_* on each of these and has a lift $\widetilde{h}_*$ agreeing with $\widetilde{h}_*$ as already defined on $\widetilde{\mathcal{N}} \cup \widetilde{\Gamma}_+ \cup \widetilde{\Gamma}_- \cup \widetilde{\mathcal{Z}}$. By the first paragraph of this proof, each $\widehat{Q}$ is carried to itself by a power h_*^p, which must be an endperiodic total translation. $\square$

8.3. **Peripheral Regions.** It remains that we extend h over components of $L \smallsetminus \mathcal{Z}$ that meet the laminations $\Lambda_\pm$. The laminations partition such a component into various sorts of components, one sort being the "peripheral regions". These are of four types. If $\gamma \in \delta\mathcal{U}$ is an immersed, essential circle, then either γ and σ are h-periodic and we have already arranged that h extends over the annulus $\widehat{Q}$ cobounded by these loops (called a peripheral region) or γ and σ are not h-periodic and σ is h-escaping. In this case, the annulus $\widehat{Q}$ that these loops cobound (again, a peripheral region) is partitioned into simply connected regions by the finitely many arcs of junctures meeting $\widehat{Q}$. By the methods of Subsection 8.1, we extend h over these components. In this case, note that all points in Q are h-escaping. If $\gamma \in \delta\mathcal{U}$ is an immersed line of the first kind, the strip $\widehat{Q}$ cobounded by γ and σ is called a peripheral region and is partitioned by arcs of junctures into bounded, simply connected regions (see Figure 4). Again the methods of Subsection 8.1 extend h over these regions and, again, all points in Q are h-escaping. Finally, if γ is of the second kind, a similar argument (carried out upstairs in the universal cover and noting that arcs of junctures of the second type already partion $\widehat{Q}$ into bounded, simply connected regions) shows that h extends over the strip $\widehat{Q}$ cobounded by γ and σ. The fact that $\widehat{Q}$ is a strip is the content of Lemma 7.29. Note that, again, all points in Q are h-escaping.

8.4. **Non-Peripheral Components.** We have defined the extension h_* on the non-rectangular components of $\mathcal{U}$ and on the nuclei of principal regions.

Let $B = B_i$ be a fundamental domain in a fundamental neighborhood of a positive end. Exactly as for $\Lambda_+ \cap K$ (see Lemma 6.8), the arcs of $\Lambda_+ \cap B$ fall into finitely many isotopy classes. If the isotopy class has more than one element, its extreme arcs are two sides of a rectangle R in B, the other two sides being subarcs of positive junctures (either of the same juncture or of consecutive ones). If the isotopy class has just one element, we call that arc a degenerate rectangle. The complement in B of the union of these finitely many (possibly degenerate) rectangles cannot have components in a positive principal region. Indeed, such a component would be in an arm of that principal region and so would be a rectangle bounded by isotopic arcs of $\Lambda_+ \cap B$. Thus, each of these complementary regions must lie in $\mathcal{U}_-$. Since $\Lambda_- \cap B = \emptyset$ and negative principal regions cannot meet B, the region lies in $\mathcal{U}_+$ also, hence in $\mathcal{U}$. Evidently it is not rectangular and so h_* is already defined on it. Each of our rectangles R contains a closed family Y of isotopic arcs of $\Gamma_+ \cap B$ and $h = h_*$ is already defined on the boundary of each component of $R \smallsetminus Y$. Thus, h can be extended continuously to h_* over each of these components. However these rectangles generally cluster in L on subarcs of Γ_+ and the whole assemblage of these extensions might not be continuous at these subarcs. But we are now assuming that the rectangles are bounded by four geodesic arcs on which h has been defined linearly as on page 20. One then uses the method in [2, Lemma 6.1] to construct the extensions over the rectangles and the continuity issues disappear. In case h_* has already been defined on some of these rectangles,

this entails redefining it there, but this is of no consequence. The extension in fundamental neighborhoods of negative ends is carried out similarly.

It remains to complete the extension in the core K. Here, the isotopy classes of $\Lambda_+ \cap K$ form one finite family of (possibly degenerate) rectangles $R_1^+, \dots, R_q^+$ bounded by extreme arcs and (possibly degenerate) segments of positive junctures, the isotopy classes of $\Lambda_- \cap K$ contributing a family $R_1^-, \dots, R_p^-$ of rectangles transverse to the first. The complementary regions to the union of these rectangles are either nuclei of principal regions, a rectangle in an arm of a principal region with one boundary arc in ∂K, or are contained in $\mathcal{U}$. In the first and third case, h_* is already defined on these regions. In the second case, the rectangle is subdivided by arcs of h-junctures into rectangles on which h_* is defined as in the above paragraph. In the intersecting rectangles, the subrectangles intersect to give rectangles with two geodesic sides in Γ_+ and two in Γ_-. There remain rectangles with two parallel geodesic sides in one of the extended laminations and lying either in one of $R_i^\pm$ or in an arm of a principal region. The nonintersecting rectangles are as in the previous paragraph. In any event, h_* is extended as in the previous paragraph over all the geodesic rectangles.

Remark. The restriction of h_* to nonperipheral components that meet the laminations are called *endperiodic pseudo-Anosov* automorphisms. There are degenerate examples in which the laminations have finitely many leaves (Gabai's "stack of chairs" [17, Example 5.1]) and one might not want to call these pseudo-Anosov. There are other difficulties about the analogy with pseudo-Anosov automorphisms of compact surfaces. While the transverse, projectively invariant measures can be defined, they often do not have full support [15]. Another notable difference is that there is no "blowdown" transforming the laminations to singular foliations. It is still true that in pseudo-Anosov components, the complementary regions to the union of the laminations have genus zero.

The proof of Theorem 8.1 is complete.

Remark that we have made no attempt to guarantee that the Handel-Miller homeomorphism h_* has any smoothness. We will speak to that question in Section 12.

9. The Core Dynamical System

Given the endperiodic map f, one can ask whether there is any uniqueness to the Handel-Miller representative h. Naively, one might hope that any two are conjugate by a homeomorphism isotopic to the identity, but because of all the "empty space" complementary to the laminations, this is unreasonable. One might hope that the laminations themselves are so conjugate, but since we only require them to be topological laminations, that might be forbiddingly difficult to prove. However, we can show that the "core dynamical system" is unique up to conjugation by an isotopy.

Definition 9.1. Let h be a Handel-Miller representative of the isotopy class of f and set $\mathcal{K} = \Lambda_+ \cap \Lambda_-$. Then the restriction of h to $\mathcal{K}$ defines the core dynamical system $h : \mathcal{K} \to \mathcal{K}$.

9.1. **Uniqueness.** Our goal is to prove the following theorem which will be critical for the application of this theory in [5].

Theorem 9.2. *Let h and h' be Handel-Miller representatives of the isotopy class of f and let $h : \mathcal{K} \to \mathcal{K}$ and $h' : \mathcal{K}' \to \mathcal{K}'$ be the associated core dynamical systems. Then there is a homeomorphism $g : L \to L$, isotopic to the identity, such that $g(\mathcal{K}) = \mathcal{K}'$ and $g^{-1} \circ h' \circ g|\mathcal{K} = h|\mathcal{K}$.*

To begin with, we "regularize" h. One replaces each arc of $\Lambda_+ \cup \Lambda_- \smallsetminus \mathcal{K}$ with the unique geodesic joining the same two endpoints in $\mathcal{K}$. In fact, if we set $\mathcal{H} = \Gamma_+ \cap \Gamma_- \cap \mathcal{Z}$, we do the same modification of $\Gamma_+ \cup \Gamma_- \cup \mathcal{Z} \smallsetminus \mathcal{H}$. We say that all of these resulting curves are "weakly piecewise geodesic". We do the same thing for the families $\Gamma'_+ \cup \Gamma'_- \cup \mathcal{Z}'$ associated to h'.

Once again, h can be redefined on all of these weakly piecewise geodesics by leaving it unchanged on $\mathcal{H}$ and extending it by linearity on each of the geodesic arcs. The proof of Theorem 8.1 only uses these geodesic arcs, hence goes through here. The new map h (which will be called a "regular Handel-Miller map") has exactly the same core dynamical system as the original one, Thus, we may assume that h and h' are both regular Handel-Miller representatives of the isotopy class of the endperiodic map f.

Proof of Theorem 9.2. We begin the construction of g in the universal cover $\widetilde{L} = \Delta$. Each point z of the lifted set $\widetilde{\mathcal{H}}$ (respectively, $\widetilde{\mathcal{H}}'$) is determined by the two curves in $\widetilde{\Gamma}_+ \cup \widetilde{\Gamma}_- \cup \widetilde{\mathcal{Z}}$ (respectively, $\widetilde{\Gamma}'_+ \cup \widetilde{\Gamma}'_- \cup \widetilde{\mathcal{Z}}'$) intersecting at z. A lift of h, h' and f can be chosen so that $\widehat{h} = \widehat{f} = \widehat{h}'$ on S^1_∞. One now knows how to define $\widetilde{g} : \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}'$ so that $\widehat{g} = \mathrm{id}$ on S^1_∞, extending $\widetilde{g}$ linearly to all of our weakly piecewise geodesics. Since the covering transformations are hyperbolic isometries, this map $\widetilde{g}$ on the families of weakly piecewise geodesic curves, commutes with the covering transformations, hence descends to a map g between the corresponding families in L. We consider components of $\widetilde{L} \smallsetminus \widetilde{\mathcal{Z}}$ and of $\widetilde{L} \smallsetminus \widetilde{\mathcal{Z}}'$ that do not meet the laminations and we correspond those $A \leftrightarrow A'$ whose boundaries are related by $\widetilde{g}$. Because $\widetilde{g}$ commutes with the covering transformations, the group of covering transformations that leave A invariant is the same as the group that leaves A' invariant. Descending down to L, we see that A and A' descend to homeomorphic surfaces. Thus, the methods in the preceding section enable us to extend this map to $g : L \to L$. Since $\widehat{g} = \mathrm{id}$ on S^1_∞, g is isotopic to the identity. By construction, $g(\mathcal{K}) = \mathcal{K}'$ and $g^{-1} \circ h' \circ g|\mathcal{K} = h|\mathcal{K}$. $\square$

9.2. **Markov Dynamics.** In this section, we show that the core dynamical system $h : \mathcal{K} \to \mathcal{K}$ is Markov, being coded by a two-ended subshift of finite type. We allow $\partial L \neq \emptyset$ and we do not reduce. We fix a core K.

Definition 9.3. A 4-*gon* R is a rectangle in K with a pair of opposite edges $\alpha, \beta \subset \Lambda_+$ and a pair of opposite edges $\delta, \gamma \subset \Lambda_-$.

Remark. We allow degenerate 4-gons. If $\alpha = \beta$, then $R = \alpha$ degenerates to an arc in Λ_+ and, if $\delta = \gamma$, R degenerates to an arc in Λ_-. If both pairs of opposite sides are equal, R degenerates to a single point. These degenerate possibiities should be kept in mind in what follows.

Definition 9.4. Let $Q' \subset K$ be a rectangle whose edges $\alpha'_Q, \beta'_Q \subset \Lambda_+$ are extreme arcs in their isotopy class (Definition 6.10). The other two edges δ'_Q, γ'_Q of Q' are arcs in positive junctures in δK. Let $Q \subset Q'$ be the 4-gon, with two sides $\alpha_Q \subset \alpha'_Q \subset \Lambda_+$ and $\beta_Q \subset \beta'_Q \subset \Lambda_+$, which is the largest such 4-gon. Let δ_Q, γ_Q be

the boundary edges of Q that are arcs in Λ_-. Let $\mathcal{Q}^+$ be the finite set of all such 4-gons.

Proposition 9.5. *If $Q \in \mathcal{Q}^+$, then $h(Q)$ completely crosses (in the positive direction) any 4-gon $Q' \in \mathcal{Q}^+$ that it meets.*

Proof. Let Q' have edges that are extreme arcs $\alpha'_Q, \beta'_Q \subset \Lambda_+$ and edges δ'_Q, γ'_Q that are subarcs of positive junctures in ∂K. The corresponding boundary arcs of Q are $\alpha_Q \subset \alpha'_Q$, $\beta_Q \subset \beta'_Q$ and $\delta_Q, \gamma_Q \subset \Lambda_-$. Suppose that $h(\delta_Q) \subset Q \smallsetminus (\delta_Q \cup \gamma_Q)$. Then $h(\delta_Q)$ and one of δ_Q, γ_Q, together with suitable subarcs of the leaves of Λ_+containing $h(\alpha'_Q)$ and $h(\beta'_Q)$, cut off a rectangle $Q'' \subset K$ such that $h^{-1}(Q'')$ can be added to Q to get a 4-gon in Q' properly containing Q. ($h^{-1}(Q'')$ is necessarily in Q' since h^{-1} pushes positive junctures away from the positive ends.) This contradicts the maximality of Q. One gets a parallel argument if $h(\gamma_Q) \subset Q \smallsetminus (\delta_Q \cup \gamma_Q)$. $\square$

In some contexts, a family of rectangles $\mathcal{Q}^+$ with the above property is called a Markov family. We will refer to $\mathcal{Q}^+$ as a *pre-Markov* family.

Definition 9.6. The family of positive *Markov 4-gons* is,

$$\mathcal{M}^+ = \{\text{components of } h(Q) \cap Q' \mid Q, Q' \in \mathcal{Q}^+\} = \{R_1^+, \dots, R_n^+\}.$$

Corollary 9.7. *If $R_i^+ \in \mathcal{M}^+$ is a component of $h(Q) \cap Q'$, then R_i^+ completely crosses Q' in the positive direction.*

Proposition 9.8. *The family $\mathcal{M}^+$ is a Markov system. That is, $h(R_i^+) \cap R_j^+$ is either empty or completely crosses R_j^+ in the positive direction and has a single component, $1 \le i, j \le n$.*

Proof. Suppose that R_i^+ is a component of $h(R) \cap R'$ with $R, R' \in \mathcal{R}^+$. Then, because h is one-one on R', it follows that $h(R_i^+) \subset h(R')$ meets every component of $h(R') \cap \left(\bigcup_{j=1}^n R_j^+\right)$ exactly once. $\square$

Remark. It is not standard to allow degenerate rectangles in Markov systems, but there is no real problem in doing so. In our situation, Markov 4-gons that degenerate to arcs are forced, for example, if there are principal regions with isolated border leaves and ones that degenerate to points occur if, after reduction, there are pieces isomorphic to Gabai's stack of chairs examples (op. cit.).

As usual, one sets up an $n \times n$ *Markov matrix* A with entries

$$A_{ij} = \begin{cases} 1, & \text{if } h(R_i^+) \cap R_j^+ \neq \emptyset, \\ 0, & \text{if } h(R_i^+) \cap R_j^+ = \emptyset. \end{cases}$$

The corresponding set $\mathcal{S}$ of *symbols* consists of all bi-infinite sequences

$$\iota = (\dots, i_{-k}, \dots, i_{-1}, i_0, i_1, \dots, i_k, \dots) \in \{1, 2, \dots, n\}^{\mathbb{Z}}$$

such that $A_{i_k, i_{k+1}} = 1$, $-\infty < k < \infty$. For such a symbol (and only for such),

$$\cdots \cap h^{-k}(R_{i_{-k}}^+) \cap \cdots \cap h^{-1}(R_{i_{-1}}^+) \cap R_{i_0}^+ \cap h(R_{i_1}^+) \cap \cdots \cap h^k(R_{i_k}^+) \cap \cdots = \zeta_\iota \neq \emptyset.$$

We would like to have that ζ_ι ranges exactly over $\mathcal{K}$ as ι ranges over $\mathcal{S}$, in which case the right-shift operator $\sigma : \mathcal{S} \to \mathcal{S}$ will be exactly conjugate to $h : \mathcal{K} \to \mathcal{K}$. This will be true if there are no principal regions, but generally many ζ_ι may be whole arcs of intersection of Λ_+ with arms of negative principal regions. We will leave it

to the reader to see this, remarking only that it is due to a basic asymmetry in the definition of $\mathcal{M}^+$ which favors the role of Λ_+. (In studying particular examples, and even for the applications of symbolic dynamics in [5, 10], it is always adequate to use $\mathcal{M}^+$. The following discussion, which does not explicitly mention the principal regions, is motivated by aesthetics.)

We define the family $\mathcal{M}^- = \{R_1^-, \dots, R_m^-\}$ of negative Markov 4-gons in perfect analogy with the definition of $\mathcal{M}^+$, obtaining then a more symmetric family $\mathcal{M} = \{R_1, \dots, R_q\}$ of Markov 4-gons, each R_ℓ being a connected component of an intersection $R_i^+ \cap R_j^-$. As we will see, the symbol set for this Markov system will exactly encode $\mathcal{K}$ and provide the desired conjugacy of $h|\mathcal{K}$ to the resulting shift map.

We will leave it to the reader to check that $\mathcal{M}$ is again a Markov system for h.

Remark. It should be remarked that our Markov 4-gons are pairwise disjoint. This is a bit stronger than the usual requirement that they merely not overlap. This eliminates the usual ambiguity in the coding, insuring that $\sigma : \mathcal{S} \to \mathcal{S}$ is conjugate to $h : \mathcal{K} \to \mathcal{K}$ and not merely semi-conjugate.

We are ready to prove the key result.

Proposition 9.9. *If $\iota = (\dots, i_{-k}, \dots, i_{-1}, i_0, i_1, \dots, i_k, \dots) \in \{1, 2, \dots, q\}^{\mathbb{Z}}$ is a symbol for $\mathcal{M}$, then the infinite intersection*

$$I_\iota^+ = R_{i_0} \cap h(R_{i_1}) \cap h^2(R_{i_2}) \cap \cdots \cap h^k(R_{i_k}) \cap \cdots$$

is an arc of $\Lambda_+ \cap R_{i_0}$ and

$$I_\iota^- = R_{i_0} \cap h^{-1}(R_{i_{-1}}) \cap h^{-2}(R_{i_{-2}}) \cap \cdots \cap h^{-k}(R_{i_{-k}}) \cap \cdots$$

is an arc of $\Lambda_- \cap R_{i_0}$. Furthermore, all such arcs are obtained in this way. Consequently, $\zeta_\iota = I_\iota^- \cap I_\iota^+ \in \mathcal{K}$ and every point of $\mathcal{K}$ is of the form ζ_ι for a unique symbol ι.

Proof. One easily sees that $\mathcal{M}^+$ and $\mathcal{M}^-$ each covers $\mathcal{K}$, hence so does $\mathcal{M}$. Consequently, the assertions about I_ι^+ and I_ι^- imply the assertion about ζ_ι. (As remarked above, the fact that each point of $\mathcal{K}$ uniquely determines its symbol is due to the fact that our Markov 4-gons are disjoint.)

If I_ι^+ is not as asserted, it must be a 4-gon with nonempty interior. Assume this and deduce a contradiction as follows. By the construction of $\mathcal{M}$, the sides of R_{i_k} in Λ_+ extend to the sides in Λ_+ of a rectangle $C_{i_k} \subset K$, the other two sides of which are arcs δ_k, γ_k in positive junctures in ∂K. Consider the set

$$P_k = h^k(C_{i_k}) \cap h^{k+1}(C_{i_{k+1}}) \cap \cdots$$

By our hypothesis, P_k is a nondegenerate rectangle with two sides δ_k', γ_k' subarcs of $h^k(\delta_k)$ and $h^k(\gamma_k)$, respectively. These are subarcs of honest junctures in fundamental neighborhoods of positive ends, hence may be tightened to geodesic segments of uniformly bounded lengths. Furthermore,

$$P_0 \subset P_1 \subset \cdots \subset P_k \subset \cdots$$

and the edges in Λ_+ of each P_k are subarcs of the corresponding sides of P_{k+1}, $0 \le k < \infty$. The increasing union of these rectangles is an infinite strip P bounded by *distinct* leaves $\lambda, \mu \in \Lambda_+$. Any lift of this strip to the universal cover is a strip with distinct boundary leaves $\widetilde{\lambda}, \widetilde{\mu} \in \widetilde{\Lambda}_+$ limiting on two pairs $\{x, y\}, \{z, w\} \subset S_\infty^1$

(Axiom 1). But the fact that δ'_k and γ'_k are of length less than one implies that $x = y$ and $z = w$, hence $\widetilde{\lambda} = \widetilde{\mu}$ (Axiom 1). This is the desired contradiction.

The assertion about I^-_ι is proven in the same way. Finally, the fact that $\mathcal{M}$ covers $\mathcal{K}$ implies that the union of all I^+_ι's also covers $\mathcal{K}$, as does the union of all I^-_ι's. Thus, all arcs of $\Lambda_+ \cap R_{i_0}$ and all arcs of $\Lambda_- \cap R_{i_0}$ are obtained as asserted, $i_0 = 1, 2, \dots, q$. □

Remark. It is possible to use the pre-Markov family $\mathcal{Q}^+ = \{Q_1, Q_2, \dots Q_p\}$ of 4-gons to produce projectively invariant measures for h on Λ_+ (and Λ_-) much as in the case of pseudo-Anosov automorphisms of compact surfaces. The following sketch follows the lead of [2, pages 95-102] and we refer the reader there for more details.

Let $B = (B_{k,j})$ be the incidence matrix, where $B_{k,j}$ is the number of components of $h(Q_k) \cap Q_j$. By the Brouwer fixed point theorem, this matrix has an eigenvector $y \neq \mathbf{0}$ with all entries nonnegative and with eigenvalue $\kappa \geq 1$. In the (typical) case that $\kappa > 1$, one obtains a transverse, projectively invariant measure μ_+ for h on Λ_+ with projective constant κ.

Now $\mathcal{Q}^+$ is also pre-Markov for h^{-1} with intersection matrix the transpose B^{T} and (left) eigenvector y^{T}. This gives a transverse, projectively invariant measure μ_- for h^{-1} on Λ_- with projective constant κ. Viewed as a projectively invariant measure for h, it has projective constant $\kappa^{-1} < 1$.

Since the eigenvectors y and y^{T} may have some zero entries, these measures will not generally have full support. See [15] for a simple example. If, however, Λ_+ is a minimal h-invariant lamination, the measures will evidently have full support.

10. The Final Axiom Set

We give here an axiomatization of Handel-Miller systems associated to an endperiodic map f which implies the axioms of Subsection 4.3 together with Axiom 9, but is now more compact. This axiomatization could not be formulated before the advent of Axiom 9, which itself could not be meaningfully posed until its truth was verified for the geodesic case.

The first four axioms are identical to those already given, but we will restate them for easy reference.

Axiom I. Λ_+ and Λ_- are mutually transverse, pseudo-geodesic laminations with all leaves noncompact and disjoint from ∂L. Furthermore, the leaves of the lifted laminations $\widetilde{\Lambda}_\pm$ are determined by their endpoints on S^1_∞.

Axiom II. The laminations Λ_+ and Λ_- are strongly closed.

Axiom III. Every leaf of $\widetilde{\Lambda}_\pm$ meets at least one leaf of $\widetilde{\Lambda}_\mp$ and can do so only in a single point.

Axiom IV. Each end of every leaf of Λ_+ (respectively, of Λ_-) passes arbitrarily near a positive (respectively, negative) end of L, but does not enter the fundamental neighborhoods of negative (respectively, positive) ends. Furthermore, if e is an end of L, every neighborhood of e meets $\Lambda_\pm$.

Axiom V. There is an endperiodic map $h : L \to L$, isotopic to f through endperiodic maps, which preserves the laminations $\Lambda_\pm$.

The assumption that h is endperiodic and isotopic to f through endperiodic maps is stronger than Axiom 9, but for the case of geodesic laminations is guaranteed by Corollary 8.2.

Our final axiom concerns h-junctures. Let $\mathcal{E}_+$ be the set of positive ends of L and $\mathcal{E}_-$ the set of negative ends. Choose a juncture J_e^0 in a fundamental neighborhood of each end e and set

$$\mathcal{X}_+ = \{h^n(J_e^0) \mid e \in \mathcal{E}_+, n \in \mathbb{Z}\}$$
$$\mathcal{X}_- = \{h^n(J_e^0) \mid e \in \mathcal{E}_-, n \in \mathbb{Z}\}.$$

Axiom VI. The choice of each J_e^0 can be made so that

(1) The lifted families $\widetilde{\Lambda}_\pm \cup \widetilde{\mathcal{X}}_\mp$ and $\widetilde{\mathcal{X}}_\pm$ are transverse and a curve in one can only meet a curve in the other in a single point;
(2) $\overline{\mathcal{X}}_\pm \smallsetminus \mathcal{X}_\pm = \Lambda_\mp$ and the curves in $\mathcal{X}_\pm$ accumulate on the leaves of $\Lambda_\mp$ locally uniformly (where we adapt Definition 4.10 to pseudo-geodesics).

Thus, $\Gamma_\pm = \Lambda_\pm \cup \mathcal{X}_\mp$ are transverse laminations and no leaf of one can intersect a leaf of the other so as to form a digon. The following definition now supersedes Definition 4.32.

Definition 10.1. The triple (Γ_+, Γ_-, h) subject to the above axioms is called a Handel-Miller system associated to the endperiodic map f. The map h itself is called a Handel-Miller representative of the isotopy class of f.

These axioms hold for the case in which the laminations are geodesic. Indeed, we have already checked this for the first four axioms, while Axiom V was established in Section 8 (Theorem 8.1 and Corollary 8.2). For Axiom VI, we take each J_e^0 to be a geodesic juncture and note that $h|\mathcal{X}_\pm$ is exactly as constructed in Subsection 4.2. For part (1) of Axiom VI, transversality is clear by construction and the fact that lifts intersect in singletons is true because they are hyperbolic geodesics. Part (2) is by construction (cf. Definition 4.7) and Proposition 4.11.

Proposition 10.2. Axioms I-VI *hold for the Handel-Miller geodesic laminations and imply the previous* Axioms 1-9.

Proof. The first assertion has been established. For the second, Axioms I-IV are identical to Axioms 1-4. Axiom 5 is obvious since $B_e^n = h^n(B_e)$. Axiom 6 is part (2) of Axiom VI. Axiom 7 is part (1) of Axiom VI. Finally, Axioms 8 and 9 follow from Axiom V and the definition of $\mathcal{X}_\pm$. □

11. Locally Stable Leaves of Finite Depth Foliations

We now turn to one of the principal goals of this paper.

Let $(M, \mathcal{F})$ be a transversely oriented, compact, C^2-foliated 3-manifold of codimension 1 and assume that $\mathcal{F}$ is taut. By doubling along $\partial_\pitchfork M$, we assume that $\partial M = \partial_\tau M$ and that the leaves of $\mathcal{F}$ have empty boundary. The results proven here extend back to the general case by restricting attention to one-half of the doubled foliated manifold. Let O be the union of the locally stable leaves (page 8), an open saturated subset of M, and let W be a component of O. It is well known (see [4, Corollary 5.2.9]) that all but finitely many components are "foliated products" and are either product foliations or have monodromy that is a total translation. We assume then, that W is one of the finitely many interesting components. An $\mathcal{F}$-transverse, 1-dimensional foliation $\mathcal{L}$, (tangent to the transverse boundary of the

original foliated manifold), gives rise to a leaf-preserving flow on $\widehat{W}$ which fixes $\partial\widehat{W}$ pointwise. Such a flow induces endperiodic monodromy $f : L \to L$ on each leaf L of $\mathcal{F}|W$ (Proposition 2.16). There is a leaf-preserving, $\mathcal{F}|W$-transverse flow on W which, induces Handel-Miller endperiodic monodromy $h : L \to L$ which is isotopic to f through endperiodic maps (Corollary 8.2). This flow is constructed in standard fashion by suspending h. Completing each flow line that approaches $\partial\widehat{W}$ by adding an ideal point produces a completion of W, homeomorphic to $\widehat{W}$ (which we identify with $\widehat{W}$), and an oriented, 1-dimensional foliation $\mathcal{L}'$ of $\widehat{W}$, transverse to $\partial\widehat{W}$, by the extended flow lines. This is generally only a C^0 foliation since h is only a homeomorphism. We remark that producing $\widehat{W}$ by the f-flow instead of the h-flow generally forces distinct h-flow lines to converge to the same point on $\partial\widehat{W}$ [15, Theorem 4.8]. We need not concern ourselves here with this rather subtle point.

If $\mathcal{F}'$ is another foliation of W, fibering W over S^1, transverse to $\mathcal{L}'$ and extending $\mathcal{F}|(M \setminus W)$ to a taut C^2 foliation of M, then a suitable parametrization of $\mathcal{L}'$ defines a flow preserving $\mathcal{F}'$, hence defines endperiodic monodromy $h' : L' \to L'$ on any leaf of $\mathcal{F}'$. Our main goal is to prove the following.

Theorem 11.1. *The homeomorphism h' is a Handel-Miller representative of its own isotopy class.*

In the case that $\widehat{W} = M$, a proof of this theorem was offered in [10, Theorem 5.8]. That proof was much too complicated, skirted some delicate issues and had an erroneous step. The development of the Handel-Miller theory in this paper makes possible a much shorter and completely rigorous proof which also works for stable components of higher depth. We proceed by defining the laminations preserved by h' and verifying the axioms of Section 10.

11.1. **Transferring Paths.** Given a path $s : [a, b] \to L$, there are countably many "transferred" paths $s' : [a, b] \to L'$, obtained by projecting locally along the leaves of $\mathcal{L}'$, which are uniquely determined by $s'(a)$. Likewise, a path $s : (-\infty, \infty) \to L$ transfers to paths $s' : (-\infty, \infty) \to L'$, each uniquely determined by any one of its values. If $s : [a, b] \to L$ is a closed loop, its transfers generally open up. However, we have the following.

Lemma 11.2. *Any transfer as above of a closed, nullhomotopic loop s on L is a closed, nullhomotopic loop s' on L'.*

Proof. If s' opens, one readily fashions a closed transversal to $\mathcal{F}'$ which is homotopic in W to s. By tautness, such a loop must be essential in M and so s is essential on L. If s' is a loop on L', it is homotopic to s in M and, again by tautness, it is nullhomotopic on L'. □

Note that, if two paths on L intersect transversely, their transfers intersect, if at all, transversely. This is because projections along $\mathcal{L}'$ define homeomorphisms of small neighborhoods in L onto such neighborhoods in L'.

We now define laminations $\Lambda'_\pm$ on L' by uniting all transfers of leaves of the Handel-Miller laminations $\Lambda_\pm$ on L. These laminations will be mutually transverse by the above remark. It is also clear that they will be h'-invariant and so Axiom V is automatic.

Remark that, by taking a large enough compact nucleus H of $\widehat{W}$, one gets an octopus decomposition for both foliations of $\widehat{W}$ such that both induce product foliations in the arms (see Subsection 2.2 for terminology and facts) and the transverse foliation $\mathcal{L}'$ of W will be tangent to $\partial_\pitchfork H$. Thus all of the "action" takes place in H, a compact sutured 3-manifold such that the restrictions of $\mathcal{F}$ and $\mathcal{F}'$ to H are depth one foliations. Thus, the proof of Theorem 11.1 reduces rather trivially to the depth one case. We work with these restricted foliations hereafter. Once again we can double along $\partial_\pitchfork H$ to eliminate boundaries of leaves. **Thus, we reduce the proof of Theorem 11.1 to the case of depth one foliations of a compact 3-manifold M with $\partial_\pitchfork M = \emptyset$ and $\partial_\tau M$ the union of the compact leaves.** The transverse 1-dimensional foliation $\mathcal{L}'$ of M_0 producing Handel-Miller monodromy on the leaves of $\mathcal{F}$ will be denoted hereafter simply by $\mathcal{L}$.

11.2. **Behavior of the Transferred Laminations at S^1_∞.** On M, fix a Riemannian metric g_M which restricts to the tangent bundle of $\mathcal{F}$ to give a leafwise hyperbolic metric $g_{\mathcal{F}}$ [3, 7, 8].

One can parametrize $\mathcal{L}$ so as to define a C^0 flow $\Phi : M \times \mathbb{R} \to M$ which is stationary on ∂M and carries leaves of $\mathcal{F}$ to leaves of $\mathcal{F}$. Fix $t \in \mathbb{R}$. If L_0 is a noncompact leaf of $\mathcal{F}$ and $L_t = \Phi_t(L_0)$, consider $\Phi_t : L_0 \to L_t$.

Let $M_0 = M \smallsetminus \partial_\tau M = M \smallsetminus \partial M$ and consider the universal cover $\widetilde{M}_0$. The foliations $\mathcal{F}$ and $\mathcal{L}$ lift to mutually transverse foliations $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{L}}$, respectively. The leaves of $\mathcal{L}$ can be parametrized to be the flow lines of an $\mathcal{F}$-preserving flow $\Phi_t : M_0 \to M_0$ and this lifts to an $\widetilde{\mathcal{F}}$-preserving flow $\widetilde{\Phi}_t : \widetilde{M}_0 \to \widetilde{M}_0$. Fix a leaf L_0 of $\mathcal{F}$ and a lift $\widetilde{L}_0 \subset \widetilde{M}_0$. This lift identifies naturally with the open unit disk Δ because the foliation is taut. One defines a homeomorphism

$$\widetilde{\Phi} : \widetilde{L}_0 \times \mathbb{R} = \Delta \times \mathbb{R} \to \widetilde{M}_0$$

by $\widetilde{\Phi}(x,t) = \widetilde{\Phi}_t(x)$. Let $\widetilde{L}_t = \widetilde{\Phi}(L \times \{t\})$ and let $\overline{L}_t$ be the union of $\widetilde{L}_t$ with its circle at infinity. Let $\overline{M}_0 = \bigcup_{-\infty<t<\infty} \overline{L}_t$. Then,

$$\widetilde{\Phi}_t : \widetilde{L}_0 = \Delta \to \widetilde{L}_t$$

is a homeomorphism, $\forall t \in \mathbb{R}$, and Theorem 3.11 shows that $\widetilde{\Phi}_t$ extends canonically to a homeomorphism,

$$\widehat{\Phi}_t : \overline{L}_0 = \mathbb{D}^2 \to \overline{L}_t, \forall t \in \mathbb{R}.$$

Thus we obtain a bijection

$$\widehat{\Phi} : \mathbb{D}^2 \times \mathbb{R} \to \overline{M}_0.$$

This bijection topologizes $\overline{M}_0$ as $\mathbb{D}^2 \times \mathbb{R}$ with the correct topology on $\widetilde{M}_0$ and on each $\overline{L}_t$. The projection $\pi : \mathbb{D}^2 \times \mathbb{R} \to \mathbb{D}^2$ identifies $\Delta = \operatorname{int} \mathbb{D}^2$ with $\widetilde{L}_0$ and $\mathbb{D}^2$ itself with the natural completion of this hyperbolic plane. The lifted laminations $\widetilde{\Lambda}_\pm$ are viewed (via π) in this copy of Δ and their completions at infinity live in $\mathbb{D}^2$.

Now select a leaf L'_0 of $\mathcal{F}'$ and one of its lifts $\widetilde{L}'_0 \subset \widetilde{M}_0$, an open disk which, generally, does not coincide with any of the factors $\Delta \times \{t\}$. Since the lifted foliation $\widetilde{\mathcal{F}}'$ is transverse to $\widetilde{\mathcal{L}}$, the leaves of which are identified with the $\mathbb{R}$-factors, the projection π identifies $\widetilde{L}'_0$ with Δ. The lifted laminations $\widetilde{\Lambda}'_\pm \subset \widetilde{L}'_0$ are the transfers along the $\mathbb{R}$-factors of $\widetilde{\Lambda}_\pm \subset \widetilde{L}_0$ and their projections to Δ under π coincide with $\widetilde{\Lambda}_\pm$. One needs to know that the circle $S^1_\infty = \partial \mathbb{D}^2$, which is the "correct" circle at

infinity for $\Delta = \widetilde{L}_0$ is also the correct one for $\Delta = \widetilde{L}'_0$. Until we know this, we will provisionally denote this circle by Σ and the correct circle at infinity by ${S^1_\infty}'$.

One notes that, while the same open disk Δ serves as the total space of the universal covers of L and L', the covering transformations are different.

Lemma 11.3. *There is an essential closed loop ρ in L which transfers to an essential closed loop ρ' in L'.*

Proof. Let J be an "honest" juncture on L. If no transfer of J to L' meets an h'-juncture on L', then any such transferred juncture must be a closed loop on L', essential by Lemma 11.2. Otherwise, find a fundamental domain $B \subset L'$ in a fundamental neighborhood of an end of L' and a transfer J' of J that meets B. By a small homotopy of J, we guarantee that $J' \pitchfork \partial B$. The arc components of $J' \cap B$ must decompose B into finitely many components, at least one of which contains an essential loop ρ'. Since ρ' is disjoint from J', any transfer ρ to L is also an essential loop. $\square$

Lemma 11.4. *The circle Σ is naturally identified with ${S^1_\infty}'$.*

Proof. Let ρ and ρ' be the closed loops of Lemma 11.3 and let R denote the set of lifts to $\Delta = \widetilde{L}_0$ of the loops $h^n(\rho)$, $n \in \mathbb{Z}$, R' the set of lifts to $\Delta = \widetilde{L}'_0$ of the loops $(h')^n(\rho')$. These two sets coincide. Their endpoints in the circle at infinity S^1_∞ for $\widetilde{L}_0$ form a dense subset X with cyclic order completely determined by the way the loops in R intersect each other. This cyclic order, viewed in Σ, is determined in the same way by R' and so the identity map on Δ carries $X \subset \Sigma$ to the correct set of endpoints of R' in ${S^1_\infty}'$ in an order-preserving way. Thus, the identity map extends to a homeomorphism $\Delta \cup \Sigma \to \Delta \cup {S^1_\infty}'$. $\square$

The proof that Axioms I and II hold for the laminations $\Lambda'_\pm$ is almost complete. It remains to show that all leaves of $\Lambda'_\pm$ are noncompact, the remaining assertions of these axioms being evident by Lemma 11.4 and the fact that these axioms hold for $\Lambda_\pm$. The noncompactness will follow from Lemma 11.9, proven in the next section.

11.3. **The First Five Axioms.**

Proposition 11.5. Axiom III *holds for $\Lambda'_\pm$.*

Proof. Since every leaf of $\Lambda_\pm$ meets a leaf of $\Lambda_\mp$, the same holds for the transferred laminations and hence for their lifts. If a leaf of $\widetilde{\Lambda}'_\pm$ meets a leaf of $\widetilde{\Lambda}'_\mp$ in two distinct points, projection gives a closed, nullhomotopic loop in L' made up of an arc in Λ'_+ and an arc in Λ'_-. By Lemma 11.2, the same holds for the laminations $\Lambda_\pm$ in L. This contradicts the assumption that Axiom III holds for $\Lambda_\pm$. $\square$

Since $h' : L' \to L'$ is an endperiodic homeomorphism, one can define the sets $\mathcal{U}'_e$ (e an end of L') and $\mathcal{U}'_\pm$ as the suitable h'-escaping sets. For instance $x \in L$ belongs to $\mathcal{U}_e$, where $e \in \mathcal{E}_+$, if the positive iterates of h' on x converge to e. On the other hand, any choice of junctures J^0_e in fundamental neighborhoods of ends can be propagated by positive and negative iterates of h' to get a system $\mathcal{X}'_\pm$ of h'-junctures which may not satisfy Axiom VI, but which can be employed in the usual way to define $\mathcal{U}'_e$ and $\mathcal{U}'_\pm$. The reader can easily check that the two definitions are equivalent.

Lemma 11.6. *The point $x \in L'$ belongs to $\mathcal{U}'_\pm$ if and only if the leaf ℓ_x of $\mathcal{L}$ through x satisfies $\ell_x \cap L \subset \mathcal{U}_\pm$.*

Proof. The negative ends of L and L' are exactly the ones that wind in on inwardly oriented components of ∂M as infinite repetitions, the positive ends likewise winding in on the outwardly oriented components of ∂M. Thus, the points $x \in \mathcal{U}_+$ are characterized by the fact that flowing them forward along $\mathcal{L}$ causes them to approach the outwardly oriented components of ∂M, the points of $\mathcal{U}_-$ being characterized analogously. Since the same characterizations hold for $\mathcal{U}'_\pm$, the claim follows. □

Since all axioms hold in L and since projections along $\mathcal{L}$ define local homeomorphisms of L to L', Corollary 4.31 for L now implies:

Corollary 11.7. *The lamination $\Lambda'_\pm$ is the frontier of $\mathcal{U}'_\mp$.*

Corollary 11.8. *No leaf of Λ'_+ (respectively, Λ'_-) meets a fundamental neighborhood of a negative (respectively, positive) end.*

This corollary is the easy part of Axiom IV. We turn to the harder part. Remark that the laminations $\Lambda'_\pm$ are closed since the laminations $\Lambda_\pm$ are closed.

Lemma 11.9. *Each leaf λ of Λ'_+ (respectively, Λ'_-) is noncompact and each end of λ passes arbitrarily near a positive (respectively, negative) end of L'.*

Proof. Otherwise, by Corollary 11.8, λ is a bounded subset of L'. For definiteness, assume that $\lambda \in \Lambda'_+$. For each integer n, let $Y_n \subset \Lambda'_+$, denote the closure in L of $(h')^n(\lambda) \cup (h')^{n-1}(\lambda) \cup \cdots \cup (h')^{n-k}(\lambda) \cup \cdots$, a closed, bounded, hence compact, Λ'_+-saturated subset of L. Since $Y_n \supset Y_{n-1}$, $\forall n$, the intersection of these sets is a compact, nonempty, saturated set $Y \subset \Lambda'_+$. Since $h(Y) = Y$, this set does not meet $\mathcal{U}'_+ \cup \mathcal{U}'_-$. Thus, its transfer back to L is a nonempty, saturated subset of Λ_+ not meeting $\mathcal{U}_+ \cup \mathcal{U}_-$. This contradicts the assumption that Axiom IV holds in L. □

As remarked in the previous section, this lemma completes the proof of the following.

Proposition 11.10. Axiom I *and* Axiom II *hold for the laminations on L'.*

Lemma 11.11. *If e' is a positive (respectively, negative) end of L', every neighborhood of e' meets Λ'_+ (respectively, Λ'_-).*

Proof. A fundamental neighborhood of e' is asymptotic to a component F of ∂M and there is an end e of L also asymptotic to F. For definiteness, assume that e' (hence e) is a positive end. Points arbitrarily near e are on a leaf of $\mathcal{L}$ meeting L' in a point arbitrarily near e'. Thus, a leaf λ of Λ_+ that passes through a point arbitrarily near e (such exists by Axiom IV) has a transfer λ' passing through a point arbitrarily near e'. □

Proposition 11.12. Axiom IV *holds for the laminations on L'.*

We recall that Axiom V holds automatically. Thus, we need only verify Axiom VI in order to complete the proof of Theorem 11.1.

11.4. **Principal Regions and Crown Sets.** In verifying Axiom VI we will need to appeal to the structure of principal regions in L'. Unfortunately, our analysis of that structure depended on the axioms in Section 4.3 and these are not all available until Axiom VI has been checked. Thus, we will need to transfer the needed information from L.

Recall that $\mathcal{P}_\pm \subset L$ is the set $L \smallsetminus (\Lambda_\pm \cup \mathcal{U}_\mp)$. Clearly, the transfer of this set is $\mathcal{P}'_\pm = L' \smallsetminus (\Lambda'_\pm \cup \mathcal{U}'_\mp)$. The components of $\mathcal{P}'_+$ (respectively, of $\mathcal{P}'_-$) will be called the positive (respectively, negative) principal regions. It is elementary that these components are the transfers of the components of $\mathcal{P}_\pm$. Nothing prevents a principal region in L from having more than one transfer to L' or distinct principal regions in L from having the same transfer. We will see, however, that $\mathcal{P}'_\pm$ has only finitely many components.

Lemma 11.13. *A simply connected principal region P in L has transferred principal regions homeomorphic to P, with nucleus a disk and the same number of arms* (*cusps*) *as P.*

Proof. The transfers are simply connected by Lemma 11.2. Under the identification of the total spaces of $\widetilde{L}$ and $\widetilde{L}'$ with Δ, the lifted laminations coincide. So the lifts of P and of its transfers also coincide. These lifts are ideal polygons, the number of cusps being exactly the number of arms downstairs. □

Consider now a principal region $P \subset L$ that is not simply connected and one of its transferred principal regions P', also not simply connected (otherwise, the reverse transfer would give simply connected principal regions by Lemma 11.13).

Lemma 11.14. *The transfer $C' \subset P'$ of a crown set $C \subset P$ has the expected structure of a crown set, with a possibly different* (*finite*) *number of arms than C. There are only finitely many transfers of C.*

Proof. Let ρ be the rim of C. The h-orbit of ρ is a finite number of essential circles. Looking at the transverse foliation $\mathcal{L}$, we see that the $\mathcal{L}$-saturation of ρ is an imbedded torus T, transverse both to $\mathcal{F}$ and $\mathcal{F}'$. Both foliations induce fibrations $T \to S^1$, hence $T \cap L'$ is a finite number of essential circles, these being exactly the transfers of ρ. The finiteness of the number of transfers C' follows. Consider a transfer $\rho' \subset C'$ of ρ. With a little thought, we see by Lemma 11.2 that the fundamental group of C' is generated by ρ'. Now, looking at the common lifts of C and C' (cf. Figure 2), one easily concludes to the desired claim. □

Lemma 11.15. *A principal region $P \subset L$ has only finitely many transfers $P' \subset L'$ and each such P' is the union of a compact, connected nucleus N', which is a transfer of the nucleus N of P, and the finitely many crown sets that are attached to N' along their rims.*

Proof. Consider the $\mathcal{L}$-saturation $R \subset M_0$ of N, a compact, connected 3-manifold bounded by finitely many tori transverse to $\mathcal{F}$ and $\mathcal{F}'$. Since $\mathcal{F}'|M_0$ is a fibration $M_0 \to S^1$, it is defined by a closed, nowhere vanishing 1-form with infinite cyclic period group. The restriction of this form to R is nonsingular with infinite cyclic period group and is transverse to ∂R. Thus, $\mathcal{F}'$ induces a fibration $R \to S^1$ and the components of $L' \cap R$ are compact fibers, all transfers to L' of the nucleus N of P. If $N' \subset P'$ is one of these, finitely many transfers of crown sets of P attach to N' along their rims and this gives the full transferred principal region P' of P. □

Evidently, N' belongs to the invariant set of h', hence will be contained in any choice of core in L'. The principal regions have finitely many arms (cusps), each beginning with a "stump" in the core (cf. the remark on page 36). These arms may escape or they may return to the core, intersecting it in rectangles.

11.5. **The Junctures and Axiom VI.** The set $\mathcal{X}'_\pm$ of h'-junctures in L' is not generally obtained by transferring $\mathcal{X}_\pm$. Indeed, examples show that a juncture in L may transfer to a non-closed curve in L'. This is a major difficulty in the proof of Axiom VI. Nevertheless, the transfers of the h-junctures are useful and we let $\mathcal{Y}_\pm$ in L' be the set of these transfers. Since Axiom VI holds in L, we conclude the following.

Lemma 11.16. *$\overline{\mathcal{Y}}_\pm \smallsetminus \mathcal{Y}_\pm = \Lambda'_\mp$ and $\mathcal{Y}_\pm$ is a (non-closed) lamination, the leaves of which accumulate on the leaves of $\Lambda'_\mp$ locally uniformly and are transverse to $\Lambda'_\pm$.*

Let $\mathcal{E}'_+$ (respectively, $\mathcal{E}'_-$) be the set of positive (respectively, negative) ends of L'. For each $e \in \mathcal{E}'_\pm$ of L', take a juncture $J^0_e \pitchfork \Lambda'_\pm$ in a fundamental neighborhood of e. The transversality can be arranged since $\Lambda'_\pm$ is covered by topological product neighborhoods. Iterating h' both forward and backward on these choices gives a preliminary choice of the families $\mathcal{X}'_\pm$. This construction and Corollary 11.7 imply part of Axiom VI.

Lemma 11.17. *The family of curves $\mathcal{X}'_\pm$ is transverse to $\Lambda'_\pm$ and $\overline{\mathcal{X}}'_\pm \smallsetminus \mathcal{X}'_\pm = \Lambda'_\mp$.*

Thus, we do not know that $\mathcal{X}'_\pm \pitchfork \mathcal{X}'_\mp$ nor that each curve of $\widetilde{\Lambda}'_\pm \cup \widetilde{\mathcal{X}}'_\mp$ meets a curve of $\overline{\mathcal{X}}'_\pm$ in at most one point. We do know that the curves of $\mathcal{X}'_\pm$ accumulate pointwise exactly on the leaves of $\Lambda'_\mp$, but we do not know that the accumulation is locally uniform. These properties can only be assured by very careful choices of the junctures J^0_e.

We can choose the core K so that the junctures J^0_e make up ∂K.

Definition 11.18. An arc α of a leaf $\Lambda'_\pm$ meeting J^0_e, $e \in \mathcal{E}'_\pm$, exactly in its endpoints, is trivial if it subtends an arc $\tau \subset J^0_e$ such that $\alpha \cup \tau$ bounds a disk in L'.

Lemma 11.19. *There are only finitely many isotopy classes of trivial arcs of leaves of Λ'_- with endpoints on J^0_e, $\forall e \in \mathcal{E}_-$. Thus, the juncture J^0_e can be chosen so that it meets no trivial arcs of Λ'_-.*

Proof. Fix $J^0_e \pitchfork \Lambda'_-$ for $e \in \mathcal{E}'_-$. Let B^0_e be the fundamental domain cobounded by J^0_e and $h^{-n_e}(J^0_e)$. Set $H_e = K \cup B^0_e$ and let $\{\alpha_k\}_{k=1}^\infty$ be trivial arcs from distinct isotopy classes of $\Lambda'_- \cap H_e$, pick an endpoint y_k of α_k and pass to a subsequence to get $y_k \to y \in J^0_e$. We can assume also that these arcs all lie on the same side of J^0_e. Thus, they either lie in $C = K$ or in $C = B^0_e$. Let α be the component of $\Lambda'_- \cap C$ with initial point y and terminal point in ∂C. This arc is is approximated uniformly well by the α_k's, k sufficiently large, hence must have both endpoints in J^0_e. Thus, for k sufficiently large, α_k is in the isotopy class of α, contrary to hypothesis. It is now a simple matter to deform J^0_e off of these finitely many trivial isotopy classes. The new juncture will again be denoted by J^0_e and we set $J^k_e = h^{kn_e}(J^0_e)$. This gives a new core K and fundamental domains B^k_e cobounded by J^k_e and J^{k+1}_e. If there is a trivial arc of $\Lambda'_- \cap (B^0_e \cup B^1_e \cup \cdots B^k_e)$, where $k \geq 1$ is minimal, then there is a trivial arc α of $\Lambda'_- \cap B^k_e$. Since Λ'_- is h'-invariant, applying a suitable negative power of h' produces a trivial arc of $\Lambda'_- \cap B^0_e$, a contradiction. □

We emphasize that this new choice of junctures may change the core K.

As we have seen, the finite family $\mathcal{P}'_\pm$ of principal regions in L' can be taken to be the transfers of $\mathcal{P}_\pm$. The crown sets transfer and the arms and their "stumps" (cf. the remark on page 36) are simply connected.

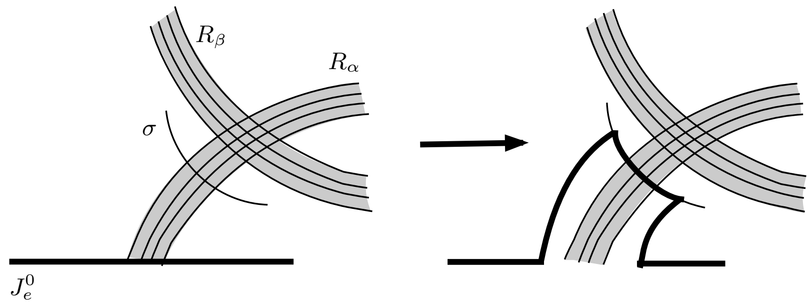

FIGURE 7. Modifying J_e^0 by a homotopy

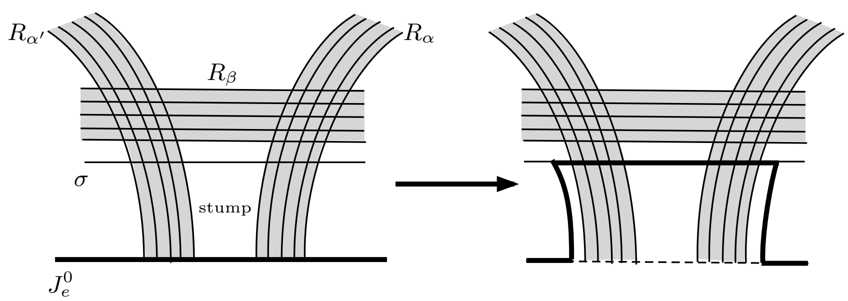

FIGURE 8. Modifying J_e^0 by a homotopy across a stump

Lemma 11.20. *As e ranges over $\mathcal{E}'_-$, the junctures J_e^0 can be chosen so that* Lemma 11.19 *holds and each J_e^0 contains finitely many subarcs of $\mathcal{Y}_-$ such that the closure of $(\mathcal{P}_- \cup \Lambda'_-) \cap J_e^0$ is contained in the union of the interiors of these subarcs.*

Proof. Fix the choice of these junctures as in Lemma 11.19 and so the new choice of core K. For each arc α of $\Lambda'_- \cap K$, let $R_\alpha \subset K$ be the minimal rectangle containing all arcs of $\Lambda'_- \cap K$ in the same isotopy class. Since there are no trivial arcs, there are only finitely many R_α's. (Lemma 6.8). There are two cases to consider: either (a) R_α intersects some R_β, where β is an arc of $\Lambda'_+ \cap K$, or (b) it does not.

In case (a), consider either of the two subrectangles R' of R_α meeting Λ'_+ only in one boundary arc and having the rest of its boundary boundary made up of one arc of $R_\alpha \cap J_e^0$ and subarcs of the sides of R_α. Since $J_e^0 \subset \mathcal{U}'_-$ and Λ'_+ is the frontier of $\mathcal{U}'_-$, R' must properly cross an arc $\sigma \subset \mathcal{Y}_-$. This is illustrated in Figure 7, along with a homotopy of J_e^0 to a juncture crossing R_α in a subarc of $\mathcal{Y}_-$. Actually, Figure 7 represents the case in which both sides of R_α border $\mathcal{U}_+$ on the outside of R_α. It is possible that one side borders a negative principal region P. In this case there is another arc α' of $\Lambda'_- \cap K$ such that the rectangles R_α, $R_{\alpha'}$ trap a stump

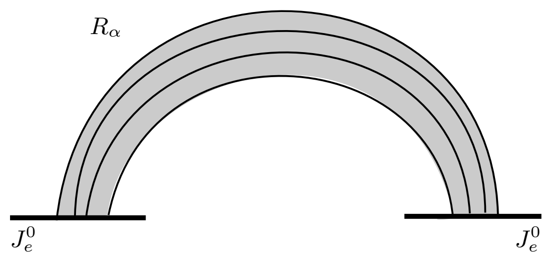

FIGURE 9. R_α does not meet Λ'_+

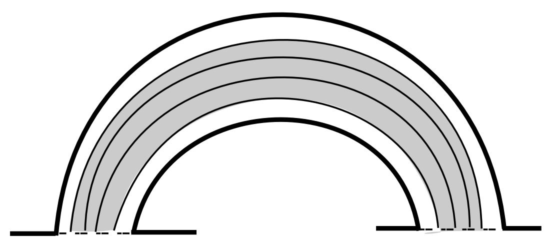

FIGURE 10. J_e^0 is modified by a "tunnel" homology

of P between them. This is illustrated in Figure 8, along with the homotopy of J_e^0 required in this case. These homotopies, repeated as often as necessary, replace every J_e^0, $e \in \mathcal{E}'_-$, occurring in case (a), with a juncture satisfying the requirements of the lemma. This procedure introduces no trivial arcs.

In case (b), R_α does not meet Λ'_+, hence lies in the connected component $\mathcal{U}'_e$ of $\mathcal{U}'_-$. Figure 9 illustrates this case and Figure 10 shows how to "tunnel" along R_α to define a homologous modification of J_e^0. This procedure may coalesce two components of J_e^0 into one, or it may split one component into two. In the latter case, if one of the components is null-homologous (bounds a compact subsurface), it should be discarded. Thus, the modified J_e^0 does not meet R_α and, again, no trivial arcs have been introduced. □

At this point, we assume that all J_e^0, $e \in \mathcal{E}'_-$, have been chosen as in Lemma 11.19 and Lemma 11.20. We define

$$\mathcal{X}'_- = \{(h')^n(J_e^0) \mid e \in \mathcal{E}'_-, n \in \mathbb{Z}\}.$$

Proposition 11.21. *The lifted families of curves $\widetilde{\mathcal{X}}'_-$ and $\widetilde{\Lambda}'_-$ are transverse and a curve in one meets a curve in the other in at most one point.*

Indeed, since Λ'_- is h'-invariant, Lemma 11.19 imples that there is no digon with one side an arc in $\mathcal{X}'_-$ and the other an arc in Λ'_-.

Proposition 11.22. $\overline{\mathcal{X}}'_- \smallsetminus \mathcal{X}'_- = \Lambda'_-$ *and the leaves of* $\mathcal{X}'_-$ *accumulate locally uniformly on the leaves of* Λ'_+.

Proof. By Lemma 11.20, each component ζ of each J^0_e is given as a head-to-tail union of compact arcs $\tau_1 \cup \sigma_1 \cup \tau_2 \cup \cdots \cup \tau_n$, with $\tau_n = \tau_1$, where each $\tau_i \subset \mathcal{U}'$ (it misses both laminations and all principal regions) and each σ_i is an arc in $\mathcal{Y}_-$, and $\zeta \cap (\Lambda'_- \cup \mathcal{P}_-)$ is contained in the union of the interiors of the σ_i's. As $n \to \infty$, the loops $(h')^n(\zeta)$ accumulate locally uniformly on the leaves of Λ'_+. Indeed, we know by Lemma 11.17 that they accumulate pointwise on these leaves. To see that the accumulation is locally uniform, let λ be a leaf of Λ_+ and let $\alpha \subset \lambda$ be a compact subarc. Let N be a suitable normal neighborhood of α such that all but finitely many components of intersections $\mathcal{Y}_- \cap N$ are cross-sections of N accumulating uniformly on α. Now components of $\mathcal{X}'_-$ accumulate pointwise on α and all but finitely many $(h')^n(\tau_i)$ are disjoint from N (the τ_i's are compact subsets of the escaping set). Thus, the components of $\mathcal{X}'_- \cap N$ are amongst those of $\mathcal{Y}_- \cap N$, accumulating uniformly on α. $\square$

At this point, we know that $\Gamma'_+ = \Lambda'_+ \cup \mathcal{X}'_-$ is an honest lamination. Now, for each $e \in \mathcal{E}'_+$, we construct J^0_e in close analogy to the construction in Lemma 11.19, but using the lamination Γ'_+ instead of Λ'_+. We then mimic the alteration in Lemma 11.20 using the lamination Λ'_+, concluding to the following.

Proposition 11.23. Axiom VI *holds on* L'.

The proof of Theorem 11.1 is complete.

12. Smoothing Handel-Miller Monodromy

We have not required that the Handel-Miller monodromy be smooth. Indeed, the construction via geodesic laminations in Section 4 has rather obvious smoothness issues. This causes some serious problems in our projected use of this theory in [5]. Here, we will prove the following.

Theorem 12.1. *The isotopy class of any endperiodic diffeomorphism f contains a Handel-Miller map that is a diffeomorphism except, perhaps, at finitely many p-pronged singularities.*

In this section, if $g : L \to L$ is endperiodic, we realize it as the monodromy of an open, foliated manifold $(W, \mathcal{F})$, where the leaves are the fibers of a smooth fibration of W over S^1, and we let $\mathcal{L}_g$ denote a 1-dimensional foliation of W, transverse to $\mathcal{F}$, having each leaf of $\mathcal{F}$ as a section, and inducing the monodromy g. The endperiodic map g will be a diffeomorphism if and only if it is possible to choose $\mathcal{L}_g$ to be smooth. In proving Theorem 12.1, we will choose h by first smoothing it in a neighborhood of the invariant set $\mathcal{I}$ and then smoothing $\mathcal{L}_h$ away from the $\mathcal{L}_h$-saturation $\mathcal{X}_h$ of $\mathcal{I}$. (The notation $\mathcal{X}_h$ should not be confused with earlier use of $\mathcal{X}_\pm$ to denote the family of positive and negative junctures. Here, $\mathcal{X}_h \subset W$ is a compact sublamination of $\mathcal{L}_h$.)

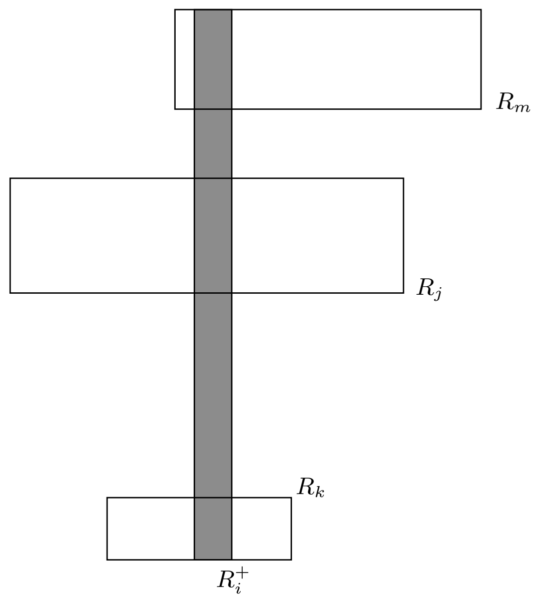

FIGURE 11. Markov Rectangles

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12.1. **Smoothing h in a neighborhood of the invariant set.** We begin with a choice of $h : L \to L$ which preserves a pair of transverse geodesic laminations Λ_+ and Λ_-. By the *meager* invariant set we mean $\mathcal{K} = \Lambda_+ \cap \Lambda_-$, a compact, totally disconnected set which is invariant under h. The full invariant set $\mathcal{I}$ is the union of $\mathcal{K}$ and finitely many compact subsurfaces with piecewise geodesic boundary.

12.1.1. *Smoothing h near the meager invariant set.* As in Subsection 9.2, we fix the Markov partition $\mathcal{M} = \{R_1, \dots, R_q\}$ for the dynamical system $h|\mathcal{K}$. This consists of finitely many convex geodesic 4-gons R_i, $1 \le i \le q$, with one pair of opposite edges (bottom and top) arcs in leaves of Λ_- and the other pair (left and right sides) in leaves of Λ_+. Recall that, for the fullest generality, we allowed for "4-gons" where one or both pairs of opposite sides degenerate to points. This nonstandard possibility causes no problems.

Let $R_i^+ = h(R_i)$, recalling that R_i^+ either completely crosses any given R_k exactly once or does not intersect it at all, as pictured in Figure 11. Similarly, set $R_i^- = h^{-1}(R_i)$. We can lay out a smooth geodesic grid on each R_k and each $R_i^\pm$ (Figure 12) so that the grids agree on overlaps. Here, the family of horizontal geodesics contains the top and bottom edges of the 4-gons and the vertical family the left and right edges.

We put one further condition on the grids. In each R_i^-, there are rectangles $A_{ij}^- = R_i^- \cap R_j$, for various values of j. There are, then, disjoint rectangles $B_{ik}^- \subset R_i^-$ which are (the closures of) the components of the complement of the union of the A_{ij}^-'s. The negative (geodesic) junctures for h will intersect each B_{ik}^- in either a countably infinite family of vertical geodesic arcs clustering exactly at each of the

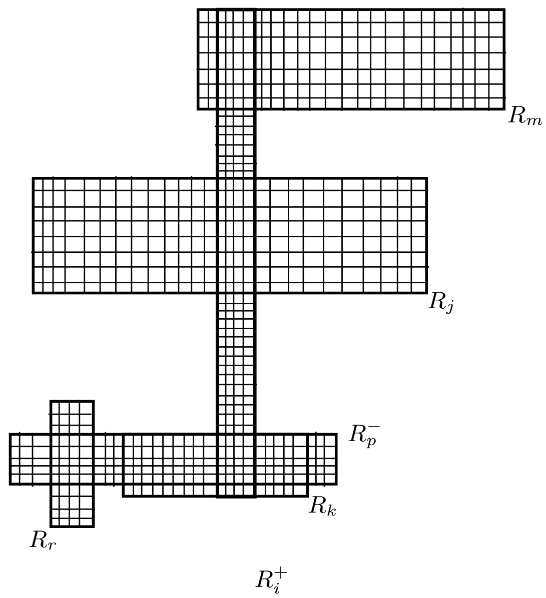

FIGURE 12. Geodesic Grids

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vertical boundaries of B_{ik}^- (this is the case that $B_{ik}^- \subset U_+$) or in an empty family (the case that B_{ik}^- lies in a principal region). The grid on each B_{ik}^- should include these vertical geodesics. Similarly, one finds rectangles $B_{ik}^+ \subset R_i^+$, complementary to the $(R_i^+ \cap R_j)$'s, and the geodesic grids in B_{ik}^+ include the horizontal arcs of intersections of positive h-junctures.

We now choose diffeomorphisms $h_i : R_i \cup R_i^- \to R_i^+ \cup R_i$ which preserve the geodesic grids, $1 \le i \le q$, and are restrictions of diffeomorphisms between slightly larger geodesic 4-gons $\widehat{R}_i \cup \widehat{R}_i^-$ and $\widehat{R}_i^+ \cup \widehat{R}_i$ containing the original 4-gons in their interiors. The geodesic grids are extended compatibly to these slightly larger 4-gons. On intersections $\widehat{R}_k \cap \widehat{R}_p^-$, we arrange that h_k and h_p agree.

In constructing the h_i's, we coordinatize the 4-gons by considering the bottom edges of the 4-gons to be the x-axes and the left edges the y-axes. For technical reasons, choose $\partial h_i/\partial x$ along the x-axes to be to be uniformly bounded below 1 and along the y-axes let $\partial h_i/\partial y$ be uniformly bounded above 1. Note that the union of the R_i's contains the meager invariant set $\mathcal{K}$, hence the union of the $\widehat{R}_i$'s contains $\mathcal{K}$ in its interior.

Consider all nonempty infinite intersections of the form

$$R_{k_0} \cap h_{k_1}(R_{k_1} \cap h_{k_2}(R_{k_2} \cap h_{k_3}(\cdots \cap h_{k_i}(R_{k_i} \cap \cdots))) \dots).$$

Because of the condition on the derivatives, this intersection will have width 0, hence will be one of the vertical geodesics in the grid on R_{k_0}. It corresponds to the symbol sequence $\kappa_+ = (k_0, k_1, k_2, \dots, k_i, \dots)$ and will be denoted by $\widetilde{\ell}_{\kappa_+}$. Generally not all sequences can occur, those that do being the "admissible" sequences for the

symbolic dynamics. This same symbol determines the arc

$$\ell_{\kappa_+} = R_{k_0} \cap h(R_{k_1} \cap h(R_{k_2} \cap h(\cdots \cap h(R_{k_i} \cap \cdots))) \dots)$$

of $\Lambda_+ \cap R_{k_0}$. In an entirely similar way, using the diffeomorphisms h_k^{-1}, symbol sequences $\kappa_- = (\dots, k_{-i}, \dots, k_{-2}, k_{-1}, k_0)$ determine horizontal geodesics $\widetilde{\ell}_{\kappa_-}$ in the grid on R_{k_0} and subarcs ℓ_{κ_-} of $\Lambda_{\kappa_-} \cap R_{k_0}$.

Since the h_k's agree on overlaps of their domains, we can unite them into a single diffeomorphism $\widetilde{h}$. Thus

$$\widetilde{\ell}_{\kappa_+} = R_{k_0} \cap \widetilde{h}(R_{k_1} \cap \widetilde{h}(R_{k_2} \cap \widetilde{h}(\cdots \cap \widetilde{h}(R_{k_i} \cap \cdots))) \dots)$$

and similarly for $\widetilde{\ell}_{\kappa_-}$.

Define $Q^- = \bigcup_{k=1}^q (R_k \cup R_k^-)$ and $Q^+ = \bigcup_{k=1}^q (R_k \cup R_k^+)$ and let $\widehat{Q}^{\pm}$ denote the enlarged versions. Then $\widetilde{h}$ extends to a diffeomorphism $\widetilde{h} : \widehat{Q}^- \to \widehat{Q}^+$. If $\widetilde{\ell}_{\kappa_+} \subset R_k$, let $\widetilde{\ell}_{\kappa_+}$ also denote the extension of this geodesic completely across $\widehat{R}_k$. Likewise, if $\widetilde{\ell}_{\kappa_-} \subset R_k$, let $\widetilde{\ell}_{\kappa_-}$ also denote the extension of this geodesic completely across $\widehat{R}_k$. Do the same for the $\ell_{\kappa_\pm}$'s. Finally, set $Q = Q^- \cup Q^+$ with associated enlargement $\widehat{Q}$.

Let $\{J_s^-\}_{-\infty<s<\infty}$ be some indexing of the segments of negative junctures occurring in the union of the B_{ik}^-'s. If $J_s^- \subset B_{ik}^-$, then $J_{si}^- = h(J_s^-)$ is a segment of negative juncture stretching vertically across R_i. Applying h again gives a segment of negative juncture $h(J_{si}^-) = J_{sii}^-$ stretching vertically across R_i^+. Then $h(J_{ssi}^- \cap R_j) = J_{siij}^-$, if nonempty, is a segment of negative juncture stretching vertically across R_j^+. Proceeding in this way, we obtain the family $\{J_{siij_1j_2\cdots j_m}^-\}$ of all negative juncture segments stretching vertically across Q. Again, use the same notations for the negative juncture segments extending vertically across $\widehat{Q}$. Similarly, successive applications of h^{-1} allow us to index the positive juncture segments stretching horizontally across Q and $\widehat{Q}$ as $\{J_{siij_1j_2\cdots j_m}^+\}$. Finally, since all $J_s^{\pm}$ belong to the geodesic grid, one uses $\widetilde{h}^{\pm 1}$ to produce corresponding arcs $\widetilde{J}_{siij_1j_2\cdots j_m}^{\pm}$ of the grid. Remark that the segments $\ell_{\kappa_\pm}$ and $J_{siij_1j_2\cdots j_m}^{\pm}$ are the components of $\Gamma_\pm \cap \widehat{Q}$ and, anticipating laminations $\widetilde{\Gamma}_\pm$, the corresponding "tilde" curves are the components of $\widetilde{\Gamma}_\pm \cap \widehat{Q}$.

One easily constructs a homeomorphism $\varphi : \widehat{Q} \to \widehat{Q}$ which carries $\Gamma_\pm \cap \widehat{Q}$ to $\widetilde{\Gamma}_\pm \cap \widehat{Q}$, preserving all the indexing. Indeed, there is a well-defined bijection $\Gamma_+ \cap \Gamma_- \cap \widehat{Q} \to \widetilde{\Gamma}_+ \cap \widetilde{\Gamma}_- \cap \widehat{Q}$ determined by the indexing. One then extends by linearity over the geodesic arcs connecting these points and then over the geodesic 4-gons formed by these arcs. Since, evidently, φ is isotopic to the identity, we extend it to a small neighborhood U of $\widehat{Q}$ so as to have support in U and be the identity elsewhere. Thus we obtain a global homeomorphism $\varphi : L \to L$, isotopic to the identity.

Observe that $\varphi \circ h \circ \varphi^{-1}$ is Handel-Miller in the same isotopy class as h, preserving the laminations $\widetilde{\Gamma}_\pm = \varphi(\Gamma_\pm)$, and agrees with $\widetilde{h}$ on $\varphi(\mathcal{K})$. Furthermore, it is easy to perturb $\widetilde{h}$ very near $\partial \widehat{Q}^-$ to a homeomorphism that agrees with $\varphi \circ h \circ \varphi^{-1}$ on $\partial \widehat{Q}^-$ and, up to parametrizations, on $\widetilde{\Gamma}_\pm$ near $\partial \widehat{Q}^-$. Replacing $\widetilde{h}$ with this new version, we obtain an isotopic endperiodic homeomorphism $\widetilde{h} : L \to L$ which agrees with $\varphi \circ h \circ \varphi^{-1}$ outside of $\widehat{Q}^-$, agrees also on $\varphi(\mathcal{K})$, preserves the laminations $\widetilde{\Gamma}_\pm$, and is a diffeomorphism on a neighborhood of $\varphi(\mathcal{K})$. The fact that $\varphi \circ h \circ \varphi^{-1}$

is Handel Miller makes it easy to check that $\widetilde{h}$, together with the laminations $\widetilde{\Gamma}_{\pm}$, satisfies Axioms I-VI of Section 10.

Theorem 12.2. *Given an endperiodic diffeomorphism f of L, the isotopy class of f contains a Handel-Miller representative that is a diffeomorphism in a neighborhood of the meager invariant set.*

12.1.2. *Smoothness in a neighborhood of the full invariant set.* The set $\mathcal{I} \smallsetminus \mathcal{K}$ presents little problem. Each component has closure the nucleus N of a principal region P, hence a compact manifold with piecewise geodesic boundary. There are three cases.

If the principal region is simply connected, N is a disk with the piecewise geodesic curve s as boundary (cf. Subsection 6.5). These fall into families cyclically permuted by h and one chooses $\widetilde{h}$ to permute them the same way, but so that $\widetilde{h}^p = \mathrm{id}$, where p is the smallest integer such that h^p fixes each vertex of s. Then $\widetilde{h}|\partial N$ is isotopic to $h|\partial N$ and one uses the isotopy to extend $\widetilde{h}$ slightly beyond N so as to match up with h and be smooth in a neighborhood of N.

If the principal region contains a single compact component C of ∂L, then N is an annulus, cobounded by C and s. Such annuli are permuted in cycles by h and one again chooses $\widetilde{h}$ as above.

In the remaining case, N is a compact surface with negative Euler characteristic. Again such surfaces are permuted in cycles by h and, in Section 8, we constructed $\widetilde{h}$ so that an appropriate power $\widetilde{h}^p$ is of Nielsen-Thurston type in $N \smallsetminus \operatorname{int} A$, where A is a union of annular collars as described in the previous paragraph. As is well understood, this is a diffeomorphism on the periodic pieces and, on the pseudo-Anosov pieces, is a diffeomorphism except at finitely many interior multi-pronged singularities. On the annuli, it can be chosen to be a diffeomorphism, a suitable power of which is the identity on each s. Once again this gives a smooth choice of h in a neighborhood of N.

Theorem 12.3. *Given an endperiodic diffeomorphism f of L, the isotopy class of f contains a Handel-Miller representative that is a diffeomorphism in a neighborhood of the invariant set $\mathcal{I}$, except at finitely many possible p-pronged singularities.*

12.2. **Smoothing h on all of L.** It is now convenient to view h as the monodromy of a fibration $W \to S^1$. We assume, by the previous section, that h is smooth on a neighborhood R of $\mathcal{I}$ (with the usual disclaimer about the finitely many p-prong singularities). We can assume that R is a compact imbedded manifold (generally not connected) with boundary, containing $\mathcal{I}$ in its interior. One produces $\mathcal{L}_h$ by the standard suspension construction. We need to be a bit careful in order to guarantee that the C^0 foliation $\mathcal{L}_h$ have as much smoothness as possible.

12.2.1. *Weak regularity of the transverse flow $\mathcal{L}_h$.* Note first, that a compactly supported isotopy replaces f with an endperiodic diffeomorphism that agrees with h on R. Thus $h \circ f^{-1}$ is the identity on R.

Select distinct leaves $L_0 = L$ and L_1 of $\mathcal{F}$ cobounding $V = L \times [0,1] \subset W$. Here, we coordinatize $L \times (-\varepsilon, \varepsilon)$ and $L \times (1-\varepsilon, 1+\varepsilon)$, small $\varepsilon > 0$, so that the interval factors are segments of leaves of $\mathcal{L}_f$ and each $L_t = L \times \{t\}$ is a leaf of $\mathcal{F}$. One can assume that trajectory of $\mathcal{L}_f$ out of $(z,1) \in L_1$ first meets L_0 in $(f(z),0)$. We need to modify $\mathcal{L}_f|V$ to $\mathcal{L}_h|V$, so that the segment s_z issuing from $(z,0) \in L_0$ terminates at $(h \circ f^{-1})(z) \in L_1$, is tangent to $\partial/\partial t$ at $(z,0)$, and $\mathcal{L}_h|L \times [0,1)$ is

smooth. Replacing $\mathcal{L}_f|V$ with this foliation produces $\mathcal{L}_h$. Evidently $\mathcal{L}_h|(W \smallsetminus L)$ will be smooth and $\mathcal{L}_h$ will be smooth at L on the positive side of that leaf.

Lemma 12.4. *There is an isotopy φ_t, $0 \le t \le 1$, of $\mathrm{id} = \varphi_0$ to $h \circ f^{-1} = \varphi_1$ such that the track of the isotopy*

$$H : L \times [0,1] \to L \times [0,1],\ H(z,t) = (\varphi_t(z), t) = (\varphi(z,t), t)$$

is smooth on $L \times [0,1) \cup R \times [0,1]$.

Proof. Let $0 < t_0 < t_1 < \cdots < t_k < \cdots \uparrow 1$ and let $\varepsilon_k > 0$ be a sequence of continuous functions on L such that $\varepsilon_k \downarrow 0$ (pointwise, hence uniformly on compact sets). Let $g_0 = \mathrm{id} = g_{t_0}$ and, for $k > 1$, let $g_k : L \times \{t_k\} \to L \times \{t_k\}$ be a diffeomorphism that ε_k-approximates $h \circ f^{-1}$. (It is well known that such approximations exist. Indeed, the proof of Munkres [20, Theorem 6.3] for 3-manifolds works equally well for surfaces.) Interpret this "closeness" via the hyperbolic metric on L. Modifying each g_k by a small isotopy compactly supported near R, we can assume that $g_{k|R=\mathrm{id}}$. By standard hyperbolic considerations, these approximating diffeomorphisms are isotopic to $h \circ f^{-1}$, hence to each other. Indeed, they are smoothly isotopic to each other since, by [12], topologically homotopic diffeomorphisms of surfaces are smoothly isotopic. This isotopy can be chosen to be the identity on R.

Choose the smooth isotopies $\varphi_k(z,t)$ of g_k to g_{k+1}, as above, parametrized on $[t_k, t_{k+1}]$. Here, $\varphi_0(z,t) = z$, $0 \le t \le t_0$. The track of the isotopy on $L \times [t_k, t_{k+1}]$ is

$$H_k(z,t) = (\varphi_k(z,t), t),\ t_k \le t \le t_{k+1}.$$

We modify the isotopy to be of the form $\widetilde{H}_k(z,t) = (\varphi_k(z, u_k(t)), t)$, where

$$u_k : [t_k, t_{k+1}] \to [t_k, t_{k+1}]$$

is smooth with $u'_k(t_k) = 0 = u'(t_{k+1})$ and $u'_k(t) > 0$, $t_k < t < t_{k+1}$. A simple computation shows that the t-derivative of H_k gives the vector $\partial/\partial t$ along $L \times \{t_k\}$ and $L \times \{t_{k+1}\}$. Thus these isotopies $\widetilde{H}_k$ fit together to give an isotopy $H(x,t) = (\varphi(z,t), t)$ of id to $g \circ f^{-1}$ on $L \times [0,1]$ which is the identity on $L \times [0, t_0]$, is smooth on $L \times [0,1)$ and is continuous at $L \times \{1\}$. Since the isotopy is the identity on R, its track $H(z,t) = z$, $0 \le t \le 1$, $z \in R$. □

Using H to define $\mathcal{L}_h|V$, we obtain the following.

Theorem 12.5. *The transverse foliation $\mathcal{L}_h$ can be chosen to be smooth on $W \smallsetminus L$ and on a neighborhood of the $\mathcal{L}_h$-saturation $\mathcal{X}_h$ of $\mathcal{I}$, except at finitely many closed orbits corresponding to p-pronged singularities of h, and to be smooth at L on its positive side. At the finitely many singular orbits, the tangent line field of $\mathcal{L}_h$ is defined and continuous.*

The continuity of the tangent line fields is guaranteed by standard local models of the flow near the orbits of p-pronged singularities (cf. [9, Appendix B]).

Finally, recall that the transverse completion $\widehat{W}$ can be formally constructed, using $\mathcal{L}_h$, in such a way that $\mathcal{L}_h$ extends tautologically to a foliation (with the same name) of $\widehat{W}$ meeting the tangential boundary $\partial_\tau \widehat{W}$ topologically transversely.

12.2.2. *Completing the smoothing of h via spiral staircases.* It will be convenient to assume that $\partial_\pitchfork M = \emptyset$ and so the leaves of $\mathcal{F}$ have empty boundary. For this, we double M, $\mathcal{F}$, $\mathcal{L}$ and $\mathcal{L}_h$ along $\partial_\pitchfork M$. Similarly, the monodromy map h is doubled. Our smoothing isotopy φ for the doubled case will be the identity on $\partial_\pitchfork \widehat{W}$, hence restricts to the desired smoothing isotopy in the nondoubled case. The advantage is that all junctures on the leaves of $\mathcal{F}|W$ can be taken to be simple closed curves. Indeed, they are finite, disjoint unions of such and can be connected by the tunneling process described in [4, page 132].

Let F be a component of $\partial_\tau \widehat{W} = \partial \widehat{W}$. For definiteness, assume that the transverse orientation of $\mathcal{F}$ is inward to $\widehat{W}$ along F. As is standard, there is a compact "spiral staircase" neighborhood $\mathcal{N}_F$ of F in $\widehat{W}$ associated to $\mathcal{F}$ and $\mathcal{L}_h$. The "roof" of $\mathcal{N}_F$ is a compact segment F' of the depth one leaf L bounded by two copies S and $h^m(S)$ of the juncture, m being the number of periodic ends of L spiraling on F. The "floor" of $\mathcal{N}_F$ is F itself. The staircase $\mathcal{N}_F$ has an "outer wall" homeomorphic to $S \times [a,b]$, where $S \times \{b\}$ is a convex corner of $\mathcal{N}_F$ and $S \times \{a\}$ is a concave corner. This wall is generally not smooth as it consists of a union of arcs in leaves of $\mathcal{L}_h$.

The union of the spiral staircases $\mathcal{N}_F$, as F ranges over the components of $\partial_\tau \widehat{W}$ that are oriented inwardly to $\widehat{W}$, will be denoted by $\mathcal{N}_-$ and the union of the floors by F_-. Analogous discussions are carried out for the components F of $\partial_\tau \widehat{W}$ oriented outwardly and the union of the corresponding spiral staircases $\mathcal{N}_F$ will be denoted by $\mathcal{N}_+$ with floor F_+. Then $\partial_\tau \widehat{W} = F_+ \cup F_-$.

As is standard, $\mathcal{L}_h$ can be parametrized as an $\mathcal{F}|W$-preserving flow Φ_t on $\widehat{W}$ which fixes $\partial \widehat{W}$ pointwise. We can choose this parametrization so that Φ_1 maps L to itself, $\Phi_1|L = h$. For each integer k, we set $\mathcal{N}^k_\pm = \Phi_k(\mathcal{N}_\pm)$. The following is quite elementary.

Lemma 12.6. *The open set*

$$\mathcal{N}^\infty_- = \bigcup_{k=0}^{\infty} \mathcal{N}^k_-$$

consists of the points $x \in \widehat{W}$ such that $\Phi_t(x)$ limits on F_- as $t \downarrow -\infty$ and the analogously defined open set $\mathcal{N}^\infty_+$ consists of the points whose forward orbits limit on F_+. Thus, $\widehat{W}$ is the disjoint union of $\mathcal{X}_h$ with $\mathcal{N}^\infty_+ \cup \mathcal{N}^\infty_-$.

Given any neighborhood U of $\mathcal{X}_h$ in W in which $\mathcal{L}_h$ is smooth, we choose $k \geq 1$ so that

$$\widehat{W} = U \cup \operatorname{int} \mathcal{N}^k_- \cup \operatorname{int} \mathcal{N}^{-k}_+.$$

Here, of course, "interior" is taken relative to $\widehat{W}$, hence does not exclude the floors. Select a nonsingular C^∞ vector field v on $\widehat{W}$ that is tangent to $\mathcal{L}_h \cap U$ and is everywhere transverse to $\mathcal{F}|\widehat{W}$. Remember that, before doubling, $\mathcal{L}_h|\partial_\pitchfork \widehat{W} = \mathcal{L}_f|\partial_\pitchfork \widehat{W}$ was smooth, so we select v to be tangent to that foliation.

Slightly modify the spiral staircases $\mathcal{N}^k_\pm$ to $\widetilde{\mathcal{N}}^k_\pm$ so that the the outer wall slants outwards so as to be transverse to v and $\mathcal{L}_h$ while remaining transverse to $\mathcal{F}$. This is slightly delicate since $\mathcal{L}_h$ is not smooth where it meets one side of L. This leaning wall will be chosen to be piecewise smooth away from that side. Details are left to the reader. Thus, $\mathcal{N}^k_\pm \subset \widetilde{\mathcal{N}}^k_\pm$ and so

$$\widehat{W} = U \cup \operatorname{int} \widetilde{\mathcal{N}}^k_\pm \cup \operatorname{int} \widetilde{\mathcal{N}}^k_\pm.$$

We emphasize that $\partial\widetilde{\mathcal{N}}^k_\pm \cap W$, again called the roof, is transverse to v and $\mathcal{L}_h$.

Lemma 12.7. *There is a homeomorphism $\varphi : W \to W$ which preserves $\mathcal{F}$ leafwise, carries $\mathcal{L}_h|W\cap\widetilde{\mathcal{N}}^k_\pm$ onto the foliation $\mathcal{L}^*$ of $W\cap\widetilde{\mathcal{N}}^k_\pm$ by v-trajectories, is supported in $\widetilde{\mathcal{N}}^k_\pm$, and is isotopic to the identity by an isotopy φ_t which preserves $\mathcal{F}$ leafwise and is supported in $\widetilde{\mathcal{N}}^k_\pm$.*

Proof. Recall that there is a transverse, holonomy-invariant measure μ for $\mathcal{F}|W$ obtained by viewing $\mathcal{F}|W$ as a fibration over the circle and lifting the Lebesgue measure. Viewed as a measure on the leaves of $\mathcal{L}_h$, this gave rise to the flow Φ_t. Similarly, restricting it to the leaves of $\mathcal{L}^*$ gives a semiflow Φ^*_t on $W\cap\widetilde{\mathcal{N}}^k_\pm$ which preserves $\mathcal{F}$ there. For each $x\in W\cap\widetilde{\mathcal{N}}^k_-$, let $\tau(x)\geq 0$ be the continuous function such that $\Phi_{\tau(x)}(x)$ is on the roof and define

$$\varphi^-(x) = \Phi^*_{-\tau(x)}(\Phi_{\tau(x)}(x)),$$

a homeomorphism of $W\cap\widetilde{\mathcal{N}}^k_-$ onto itself that extends by the identity on the rest of W, carries each leaf of $\mathcal{F}$ to itself, and carries $\mathcal{L}_h|W\cap\widetilde{\mathcal{N}}^k_-$ onto $\mathcal{L}^*|W\cap\widetilde{\mathcal{N}}^k_\pm$. Replace $\mathcal{L}_h$ by the foliation $\mathcal{L}^*_h$ integral to v. This agrees with $\mathcal{L}^*$ in $W\cap\widetilde{\mathcal{N}}^k_\pm$ and with $\mathcal{L}_h$ outside of that set. Note that, $\mathcal{X}^*_h = \mathcal{X}_h$.

We construct an appropriate isotopy φ^-_s, $0\leq s\leq 1$, with $\varphi^-_0 = \mathrm{id}$ and $\varphi^-_1 = \varphi^-$. In $W\cap\widetilde{\mathcal{N}}^k_-$, define

$$\varphi^-_s(x) = \Phi^*_{-s\tau(x)}(\Phi_{s\tau(x)}(x)).$$

This is supported in $W\cap\widetilde{\mathcal{N}}^k_-$, hence extends by the identity on the rest of W. It preserves $\mathcal{F}$ leafwise.

Repeat this argument in $W\cap\widetilde{\mathcal{N}}^k_+$, where now $\mathcal{L}^*_h$ replaces $\mathcal{L}_h$, to obtain φ^+_s. One might fear a conflict at points $x\in\widetilde{\mathcal{N}}^k_-\cap\widetilde{\mathcal{N}}^k_-$, but such a point lies on an $\mathcal{L}^*_h$ trajectory that connects F_- to F_+. In $\widetilde{\mathcal{N}}^k_+\cap\widetilde{\mathcal{N}}^k_-$, this trajectory already agrees with $\mathcal{L}^*$, hence $\varphi^+_s(x) = x$, $0\leq s\leq 1$. We set $\varphi_s = \varphi^+_s\circ\varphi^-_s$. Note also that $\varphi_s\equiv\mathrm{id}$ on the original $\partial_\pitchfork\widehat{W}$ before doubling. □

Since φ_s preserves L, we can replace the monodromy h with the equivalent monodromy $\varphi\circ h\circ\varphi^{-1}$ and the foliation $\varphi(\mathcal{L}_h) = \mathcal{L}^*_h$, smooth except at finitely many closed orbits and integral to the vector field v, can now serve as $\mathcal{L}_{\varphi\circ h\circ\varphi^{-1}}$, with $\mathcal{X}_{\varphi\circ h\circ\varphi^{-1}} = \mathcal{X}^*_h = \mathcal{X}_h$. We have completed the proof of Theorem12.1.

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Department of Mathematics, St. Louis University, St. Louis, MO 63103
E-mail address: CANTWELLJC@SLU.EDU

Department of Mathematics, Washington University, St. Louis, MO 63130
E-mail address: LC@MATH.WUSTL.EDU